Practice Exam 3 — Fundamentals of Calculus, ch. 1-5

1 A falling rock has a height (in meters) as a function of time (in seconds) given by $h(t) = pt^2 + qt + r$, where p, q, and r are constants.
(a) Infer the units of p, q, and r.
(b) Find the velocity $h'(t)$ and acceleration $h''(t)$.
(c) Show that your answers to part b have units that make sense.

2 (a) State the geometrical interpretation of the size of the second derivative, i.e., what a large second derivative tells you compared to a small second derivative (assuming the same sign).
(b) Let $y = \sqrt{1+x^2}$. Describe the symmetry of this function, i.e., is it odd, even, or neither?
(c) Find y'. Describe its symmetry, and use this as a check on your answer.
(d) Find y'' and again describe the symmetry and use it as a check.
(e) For the original function y, find the range and domain, describe any extrema, and sketch the graph. Hint: there are two slanted asymptotes, which can be found by considering the behavior of the dominant term for large positive and negative x.
(f) Where is y'' large, and where is it small? Check that makes sense in terms of your geometrical interpretation from part a and the graph you sketched in part e.

3 Evaluate the following limits. Explain your reasoning, but it is not necessary to give a formal $\epsilon - \delta$ proof or to appeal explicitly to properties of the limit if your reasoning is clear and convincing otherwise. In some cases you may find it helpful to examine the behavior of the given expression for particular values of x for which it is easy to evaluate without a calculator; this is *not* sufficient for full credit, but is worth partial credit if it leads you to a correct guess. If a limit is infinite or undefined, say so; give the most specific possible description, including the sign for infinite limits ($+\infty$ or $-\infty$). The notation $\lim_{x\searrow a}$ means the same thing as $\lim_{x\to a^+}$, and similarly $\lim_{x\nearrow a}$ means $\lim_{x\to a^-}$.

(a) $\lim\limits_{x\to\infty} \dfrac{7+8x+9x^2}{10+11x+12x^2}$

(b) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{1+100x^6}}{x^3}$

(c) $\lim\limits_{x\searrow 0} \dfrac{\sqrt{x}}{x}$

(d) $\lim\limits_{x\to 0} \dfrac{x^2+x^3}{x+x^4}$

(e) $\lim\limits_{x\nearrow 0} x^{-5}$

4 Let the function f be defined by $f(x) = e^{x^3}$. Find any and all local maxima, local minima, and points of inflection of this function.

5 Differentiate the following expressions with respect to x.

(a) $\sin(2x)$

(b) $3e^{5x} + 2$

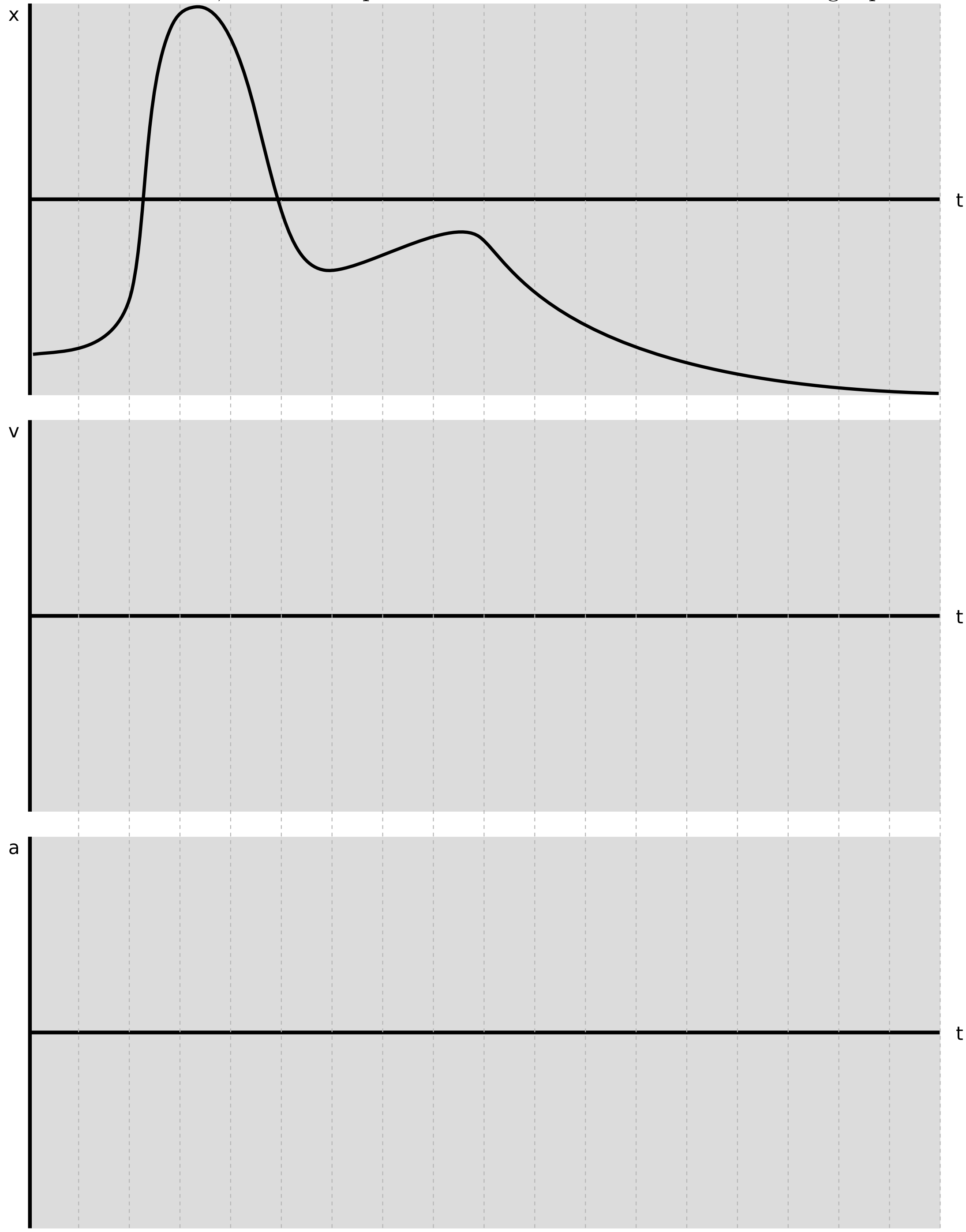
(c) $(\cos x + \sin x)/2$

(d) $-\cos(-x)$

(e) e^{x^2}

——————> *Turn the page.*

6 The top graph shows the position of a honeybee as a function of time. In the middle panel, sketch the honeybee's velocity $v = \mathrm{d}x/\mathrm{d}t$ as a function of time, and in the bottom panel its acceleration $a = \mathrm{d}^2x/\mathrm{d}t^2$. Hint: I find that it helps to start by plotting points where the derivative is zero, and also points where the derivative has a large positive or negative value.



7 Each of the following expressions defines a function of x. Find the derivative of each function at $x = 0$, simplifying your result as much as possible. The letters a and b stand for constants.

(a) $\ln(a - bx)$ (b) $\frac{b}{\pi}\sin\left(\pi e^{x/a}\right)$

(c) $\frac{b}{1 + \ln(1 + x/a)}$ (d) $\sin\sin\sin\sin(x^2)$

8 The general idea of the intermediate value theorem is that if a function varies from value y_1 to value y_2, then it must also have a point where its value is y_3, if y_3 is between y_1 and y_2. List two additional assumptions that are necessary for the theorem to be valid, and demonstrate using counterexamples that it fails if these assumptions are omitted.

Answer to problem 1

(a) Because h has units of meters, all three terms on the right must also have units of meters. That means r has units of meters, q m/s, and p m/s^2.

(b) Differentiation gives

$$h'(t) = 2pt + q,$$

and a second differentiation results in

$$h''(t) = 2p.$$

(c) Another way of writing the velocity $h'(t)$ is $\mathrm{d}h/\mathrm{d}t$, which shows that its units should be m/s. These do check out against the units of both terms on the right:

$$\frac{\mathrm{m}}{\mathrm{s}} = \frac{\mathrm{m}}{\mathrm{s}^2} \cdot \mathrm{s} + \frac{\mathrm{m}}{\mathrm{s}}$$

We expect the acceleration $h''(t) = \mathrm{d}^2h/\mathrm{d}t^2$ to have units of m/s^2, and these units also check out, since the right-hand side is $2p$, and p has units of m/s^2.

Answer to problem 2

(a) The second derivative measures the curvature of the graph.

(b) It's even, because x enters into the expression only as x^2, and therefore flipping of the sign of x won't change the output of the function.

(c)

$$\begin{aligned} y' &= \left[\sqrt{1+x^2}\right]' \\ &= \left[(1+x^2)^{1/2}\right]' \\ &= \frac{1}{2}(1+x^2)^{-1/2}(2x) \\ &= x(1+x^2)^{-1/2} \end{aligned}$$

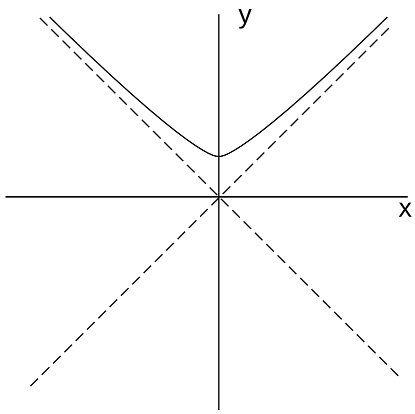
It's odd, because flipping the sign of x will flip the sign of the factor of x in front, but won't affect the sign of the factor that has x^2 inside. This makes sense, because y is even, and the derivative of an even function should be odd.

(d) Using the product rule, we have

$$\begin{aligned} y'' &= \left[x(1+x^2)^{-1/2}\right]' \\ &= (1+x^2)^{-1/2} + x \cdot \left(-\frac{1}{2}\right)(1+x^2)^{-3/2}(2x) \\ &= (1+x^2)^{-1/2} - x^2(1+x^2)^{-3/2} \\ &= (1+x^2)^{-3/2} \end{aligned}$$

This is even, and that makes sense, because it's the derivative of an odd function.

(e) The domain is the whole real line, since the formula defining the function is well defined for any real x. This is a smooth function, so its local extrema occur where the derivative is zero. The derivative is zero only at $x = 0$, which is clearly a minimum, at $y = 1$. Since the function grows without bound for large positive and negative x, its range is $[1, \infty)$. The local minimum at $(0, 1)$ is also a global minimum. For large x, the x^2 term dominates, so we have $y \approx \sqrt{x^2} = |x|$. This tells us that we have two oblique asymptotes, which are the lines $y = \pm x$.



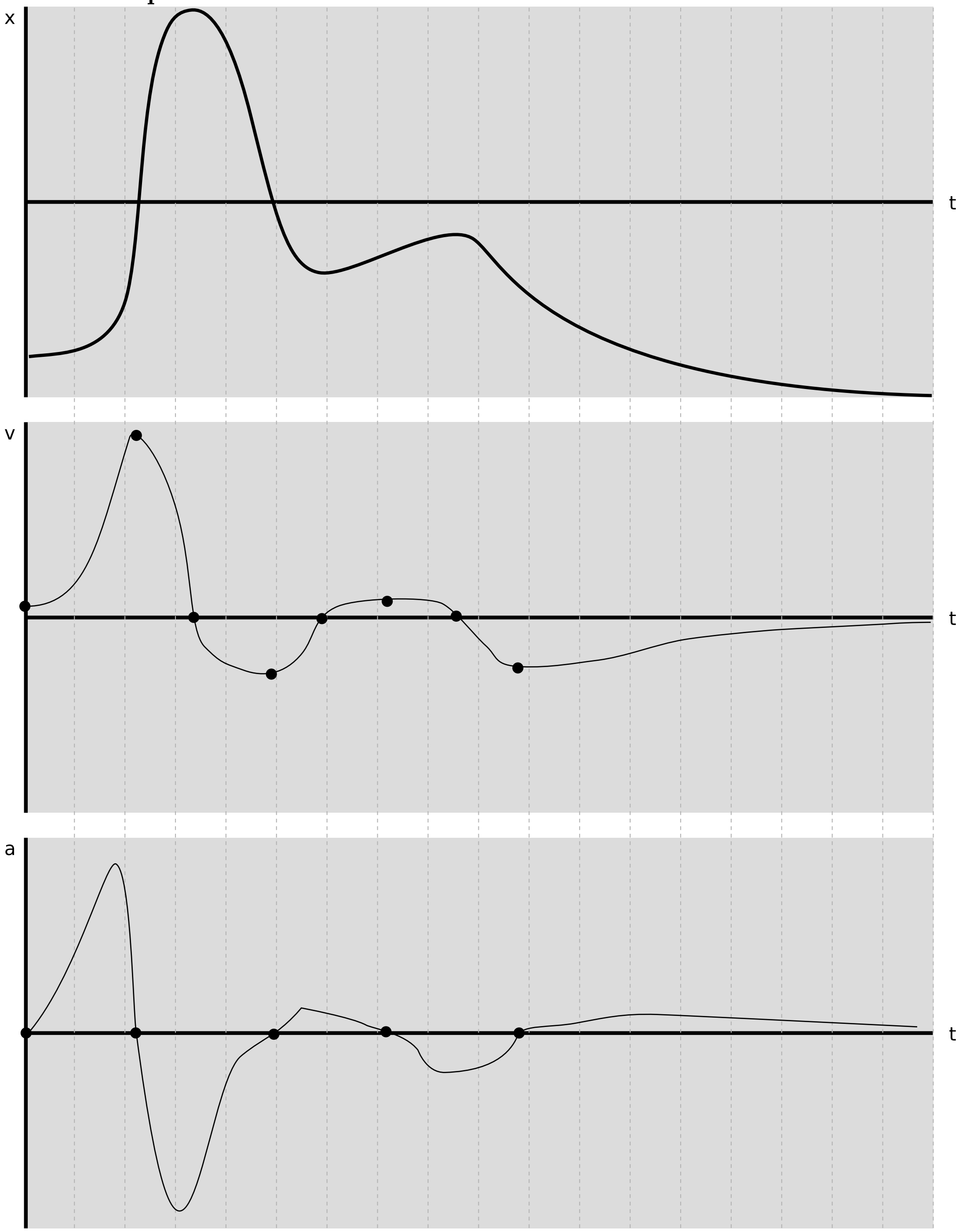
(f) The function $y'' = (1+x^2)^{-3/2}$ is built out of $1+x^2$, which is an increasing function of the absolute value of x, and $(\ldots)^{-3/2}$, which is a decreasing function of its input. Therefore y'' decreases as the absolute value of x increases. It's largest at $x=0$, and smallest for large values of x. That makes sense, because the graph is highly curved near $x=0$, but has almost no curvature where it approaches the asymptotes.

Answer to problem 3
(a) For large x, the x^2 terms dominate, so the limit is $9/12 = 3/4$.
(b) For large x, the x^6 term inside the square root dominates, so the square root becomes approximately $10|x|^3$. The limit is -10.
(c) Simplifying the expression gives $\pm x^{-1/2}$. From the right, the function is positive, and the right-hand limit is $+\infty$.
(d) For small x, the dominant terms are x^2 on the top and x on the bottom. The limit of our expression is therefore the same as the limit of $x^2/x = x$, or zero.
(e) As x approaches 0 from below, the expression x^{-5} is negative and grows without bound. The limit is $-\infty$.

Answer to problem 4
This is the composition of two differentiable functions, so it's differentiable, and therefore its local extrema occur where its derivative is zero. The derivative is $f'(x) = 3x^2e^{x^3}$, which is zero only at $x=0$. The second derivative is $f''(x) = (9x^4+6x)e^{x^3}$, which equals zero at $x=0$. The second derivative test is inconclusive, so we need to figure out whether the derivative changes sign at $x=0$. Since our expression for $f'(x)$ is the product of positive factors, it's always positive. If f' never changes signs, then it can't change signs at $x=0$, and $x=0$ must be an inflection point.

Answer to problem 5
(a) $[\sin(2x)]' = 2\cos(2x)$ (chain rule)
(b) $\left[3e^{5x}+2\right]' = 15e^{5x}$ (chain rule)
(c) $(-\sin x + \cos x)/2$
(d) $(-1)(-\sin(-x))(-1) = -\sin(-x)$
(e) $2xe^{x^2}$ (chain rule)

Answer to problem 6



Answer to problem 7

(a) By the chain rule, the derivative is $-b/(a-bx)$, which at $x=0$ gives $-b/a$.

(b) The chain rule gives $(b/\pi)\cos(\pi e^{x/a})\pi e^{x/a}(1/a) = (b/a)\cos(\pi e^{x/a})$. At zero this equals $(b/a)\cos\pi = -b/a$.

(c) $\left[b(1+\ln(1+x/a))^{-1}\right]' = -b(1+\ln(1+x/a))^{-2}(1+x/a)^{-1}(1/a) = -(b/a)(1+\ln(1+x/a))^{-2}(1+x/a)^{-1}$. At zero this gives $-b/a$.

(d) This has the form $f(x^2)$, so its derivative is $2xf'(x^2)$, and this equals zero at $x=0$.

Answer to problem 8

It has to be a continuous function. Otherwise we could have, for example, the function $f(x)$ that is 0 when $x<0$ and 1 when $x>1$. This function goes from $y_1=0$ at $x=-1$ to $y_2=1$ at $x=1$, but it never achieves the value $y_3=1/2$.

It has to be a real function. As a counterexample, suppose that $g(x)=x^2$ is defined on the rational numbers. Then g never passes through the value 7, because the square root of 7 is irrational.