Practice Exam 4 — Fundamentals of Calculus, ch. 1-7

1 Use L'Hôpital's rule to evaluate the following limits. If a limit is infinite or undefined, say so; give the most specific possible description, including the sign for infinite limits ($+\infty$ or $-\infty$).

(a) $\lim_{x\to 0} \dfrac{\sin x + x}{e^x - 1}$

(b) $\lim_{x\to 0} \dfrac{e^x + e^{-x} - 2}{\cos x - 1}$

(c) $\lim_{x\to \infty} \dfrac{\ln x}{x}$

(d) $\lim_{x\to 1} \dfrac{\ln x}{x-1}$

2 The equation $\cos x = x^3$ has a single solution, which is near $x = 1$. Use Newton's method to approximate this solution to three decimal places.

3 The dice in a certain casino are cubes with sides of length ℓ. They get used heavily, which causes them to wear down at a rate $\mathrm{d}\ell/\mathrm{d}t$. Find the rate at which they lose volume $\mathrm{d}V/\mathrm{d}t$, in terms of ℓ and $\mathrm{d}\ell/\mathrm{d}t$.

4 Evaluate the following differentials. Example: The differential of x^2 is $\mathrm{d}(x^2) = 2x\mathrm{d}x$.

(a) $3x + 4$

(b) x^3

(c) xe^x

(d) $\cos(x^2)$

(e) $(1 - \sqrt{x})^{-1}$

5 The volume of a pyramid with a square base is $(1/3)b^2h$, where h is the height and b is the length of one side of the base. The gardener at a miniature golf course is maintaining a bush in the shape of a pyramid. He decides it will look more artistic if he lets the base grow a little bit while keeping the same height.

(a) Evaluate the differential of the volume, treating h as a constant and b as the only variable.

(b) Suppose that initially the base is 2.00 m and the height is 3.00 m. Using your answer from part a, approximate the increase in volume if the base is allowed to grow by 0.10 m. You should be able to get the result without resorting to your calculator. Do not resort to brute-force calculation of the new volume to get an exact result; the point of the problem is to use the differential as an approximation.

6 Let x and y be related by $xe^y + ye^x = 0$. The graph of this relation passes through the origin. Use implicit differentiation to find its slope at the origin.

Answer to problem 1

(a) The top and bottom both approach zero, so L'Hôpital's rule applies. Differentiation of the top and bottom gives

$$\lim_{x\to 0} \frac{\cos x + 1}{e^x},$$

which equals 2 when we plug in.

(b) The top and bottom both approach zero, so L'Hôpital's rule applies. Differentiation of the top and bottom gives

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{-\sin x}.$$

This is still the indeterminate form 0/0. Going again, we get

$$\lim_{x\to 0} \frac{e^x + e^{-x}}{-\cos x},$$

which equals -2 when we plug in.

(c) This is the indeterminate form ∞/∞. Differentiating the top and bottom gives

$$\lim_{x\to \infty} \frac{1/x}{1},$$

and plugging in results in 0. This makes sense, because $\ln x$ doesn't grow as fast as x.

(d) This is the indeterminate form 0/0. Differentiating the top and bottom gives

$$\lim_{x\to 1} \frac{1/x}{1},$$

and plugging in gives 1. This makes sense, because the limit is the same one we would have used in calculating the derivative of $\ln x$ at $x = 1$, and that derivative equals 1.

Answer to problem 2

Let $y = x^3 - \cos x$. We're searching for a solution of $y = 0$. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} \approx \frac{\Delta y}{\Delta x},$$

and solving for Δx gives

$$\Delta x \approx \frac{\Delta y}{\mathrm{d}y/\mathrm{d}x} = \frac{\Delta y}{3x^2 + \sin x}.$$

Evaluating our function at the initial guess of $x = 1$, we have $y = 0.46$, and plugging in to the equation above gives $\Delta x = -0.12$, so our new estimate is $x = 0.88$. This gives $y = 0.044$, which is better. A second iteration gives $\Delta x = -0.0143$ and $x = 0.866$. One more iteration gives $x = 0.865$, which is good to three decimal places.

Answer to problem 3

We have $V = \ell^3$, so

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}V}{\mathrm{d}\ell}\frac{\mathrm{d}\ell}{\mathrm{d}t} \\ &= 3\ell^2\frac{\mathrm{d}\ell}{\mathrm{d}t}\end{aligned}$$

Answer to problem 4

(a) $3\mathrm{d}x$

(b) $3x^2\mathrm{d}x$ (power rule)

(c) $e^x\mathrm{d}x + xe^x\mathrm{d}x$ (product rule)

(d) $-2x\sin(x^2)\mathrm{d}x$ (chain rule)

(e) $\mathrm{d}\left((1-\sqrt{x})^{-1}\right) = -(1-\sqrt{x})^{-2}\left(-\frac{1}{2}x^{-1/2}\right)\mathrm{d}x = \frac{1}{2}x^{-1/2}(1-\sqrt{x})^{-2}\mathrm{d}x$

Answer to problem 5
(a) $\mathrm{d}V = (2/3)bh\mathrm{d}b$.
(c) $\Delta V \approx (2/3)bh\Delta b = 0.40$ m.

Answer to problem 6
Taking differentials gives

$$e^y\mathrm{d}x + xe^y\mathrm{d}y + e^x\mathrm{d}y + ye^x\mathrm{d}x = 0.$$

Separating the $\mathrm{d}x$ and $\mathrm{d}y$ terms, we find

$$(e^y + ye^x)\mathrm{d}x = -(xe^y + e^x)\mathrm{d}y,$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^y + ye^x}{xe^y + e^x},$$

and plugging in $x = 0$ and $y = 0$ gives $\mathrm{d}y/\mathrm{d}x = -1$.