gets boosted to \((E/2)S\). In frame B, as in A, O retains the same speed after emission of the light. But observers in frames A and B disagree on how much energy O has lost, the discrepancy being

\[ E \left[ \frac{1}{2} (S + S^{-1}) - 1 \right]. \]

Let’s consider the case where B’s velocity relative to A is small. Expanding the above expression in a Taylor series in \(v\), the discrepancy in O’s energy loss is approximately

\[ \frac{1}{2} Ev^2/c^2. \]

The interpretation is that when O reduced its energy by \(E\) in order to make the light rays, it reduced its mass from \(m_o\) to \(m_o - m\), where \(m = E/c^2\). Rearranging factors, we have Einstein’s famous

\[ E = mc^2. \]

This derivation entailed an approximation, and redoing it without the approximation entails some complexity.\(^1\) It turns out, however, to be valid in general.

We find that mass is not simply a built-in property of the particles that make up an object, with the object’s mass being the sum of the masses of its particles. Rather, mass and energy are equivalent, so that if the experiment of figure j is carried out with a sufficiently precise balance, the reading will drop because of the mass equivalent of the energy emitted as light.

The equation \(E = mc^2\) tells us how much energy is equivalent to how much mass: the conversion factor is the square of the speed of light, \(c\). Since \(c\) a big number, you get a really really big number when you multiply it by itself to get \(c^2\). This means that even a small amount of mass is equivalent to a very large amount of energy. Conversely, an ordinary amount of energy corresponds to an extremely small mass, and this is why nobody detected the non-null result of experiments like the one in figure j hundreds of years ago.

The big event here is mass-energy equivalence, but we can also harvest a result for the energy of a material particle moving at a certain speed. Plugging in \(S = \sqrt{(1 + v)/(1 - v)}\) to the equation above for the energy discrepancy of object O between frames A and B, we find \(m(\gamma - 1)c^2\). This is the difference between O’s energy in frame B and its energy when it is at rest, but since mass and energy are equivalent, we assign it energy \(mc^2\) when it is at rest. The result is that the energy is

\[ E = mc^2. \]

Electron-positron annihilation example 3

Natural radioactivity in the earth produces positrons, which are like electrons but have the opposite charge. A form of antimatter, positrons annihilate with electrons to produce gamma rays, a form of high-frequency light. Such a process would have been considered impossible before Einstein, because conservation of mass and energy were believed to be separate principles, and this process eliminates 100% of the original mass. The amount of energy produced by annihilating 1 kg of matter with 1 kg of antimatter is

\[
E = mc^2 \\
= (2 \text{ kg}) \left(3.0 \times 10^8 \text{ m/s}\right)^2 \\
= 2 \times 10^{17} \text{ J},
\]

which is on the same order of magnitude as a day’s energy consumption for the entire world’s population!

Positron annihilation forms the basis for the medical imaging technique called a PET (positron emission tomography) scan, in which a positron-emitting chemical is injected into the patient and mapped by the emission of gamma rays from the parts of the body where it accumulates.

A rusting nail example 4

▷ An iron nail is left in a cup of water until it turns entirely to rust. The energy released is about 0.5 MJ. In theory, would a sufficiently precise scale register a change in mass? If so, how much?

▷ The energy will appear as heat, which will be lost to the environment. The total mass-energy of the cup, water, and iron will indeed be lessened by 0.5 MJ. (If it had been perfectly insulated, there would have been no change, since the heat energy would have been trapped in the cup.) The speed of light is \(c = 3 \times 10^8 \text{ meters per second}\), so converting to mass units, we have

\[
m = \frac{E}{c^2} \\
= \frac{0.5 \times 10^6 \text{ J}}{(3 \times 10^8 \text{ m/s})^2} \\
= 6 \times 10^{-12} \text{ kilograms}.
\]

The change in mass is too small to measure with any practical technique. This is because the square of the speed of light is such a large number.

Gravity bending light example 5

Gravity is a universal attraction between things that have mass, and since the energy in a beam of light is equivalent to some very small amount of mass, we expect that light will be affected
by gravity, although the effect should be very small. The first important experimental confirmation of relativity came in 1919 when stars next to the sun during a solar eclipse were observed to have shifted a little from their ordinary position. (If there was no eclipse, the glare of the sun would prevent the stars from being observed.) Starlight had been deflected by the sun's gravity. Figure m is a photographic negative, so the circle that appears bright is actually the dark face of the moon, and the dark area is really the bright corona of the sun. The stars, marked by lines above and below them, appeared at positions slightly different than their normal ones.

Example 5. Black holes example 6
A star with sufficiently strong gravity can prevent light from leaving. Quite a few black holes have been detected via their gravitational forces on neighboring stars or clouds of gas and dust.
**Summary**

**Selected vocabulary**

potential energy  the energy having to do with the distance between two objects that interact via a noncontact force

**Notation**

PE . . . . . . potential energy

**Other terminology and notation**

$U$ or $V$ . . . . . symbols used for potential energy in the scientific literature and in most advanced textbooks

**Summary**

Historically, the energy concept was only invented to include a few phenomena, but it was later generalized more and more to apply to new situations, for example nuclear reactions. This generalizing process resulted in an undesirably long list of types of energy, each of which apparently behaved according to its own rules.

The first step in simplifying the picture came with the realization that heat was a form of random motion on the atomic level, i.e., heat was nothing more than the kinetic energy of atoms.

A second and even greater simplification was achieved with the realization that all the other apparently mysterious forms of energy actually had to do with changing the distances between atoms (or similar processes in nuclei). This type of energy, which relates to the distance between objects that interact via a force, is therefore of great importance. We call it potential energy.

Most of the important ideas about potential energy can be understood by studying the example of gravitational potential energy. The change in an object’s gravitational potential energy is given by

$$\Delta PE_{grav} = -F_{grav}\Delta y,$$

[if $F_{grav}$ is constant, i.e., the motion is all near the Earth’s surface]

The most important thing to understand about potential energy is that there is no unambiguous way to define it in an absolute sense. The only thing that everyone can agree on is how much the potential energy has changed from one moment in time to some later moment in time.

An implication of Einstein’s theory of special relativity is that mass and energy are equivalent, as expressed by the famous $E = mc^2$. The energy of a material object is given by $E = mc^2$. 

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Problems

Key
√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 A ball rolls up a ramp, turns around, and comes back down. When does it have the greatest gravitational potential energy? The greatest kinetic energy? [Based on a problem by Serway and Faughn.]

2 Anya and Ivan lean over a balcony side by side. Anya throws a penny downward with an initial speed of 5 m/s. Ivan throws a penny upward with the same speed. Both pennies end up on the ground below. Compare their kinetic energies and velocities on impact.

3 Can gravitational potential energy ever be negative? Note that the question refers to $PE$, not $\Delta PE$, so that you must think about how the choice of a reference level comes into play. [Based on a problem by Serway and Faughn.]

4 (a) You release a magnet on a tabletop near a big piece of iron, and the magnet slides across the table to the iron. Does the magnetic potential energy increase, or decrease? Explain.
   (b) Suppose instead that you have two repelling magnets. You give them an initial push towards each other, so they decelerate while approaching each other. Does the magnetic potential energy increase or decrease? Explain.

5 Let $E_b$ be the energy required to boil one kg of water. (a) Find an equation for the minimum height from which a bucket of water must be dropped if the energy released on impact is to vaporize it. Assume that all the heat goes into the water, not into the dirt it strikes, and ignore the relatively small amount of energy required to heat the water from room temperature to $100^\circ$C. [Numerical check, not for credit: Plugging in $E_b = 2.3$ MJ/kg should give a result of 230 km.]
   (b) Show that the units of your answer in part a come out right based on the units given for $E_b$.

6 A grasshopper with a mass of 110 mg falls from rest from a height of 310 cm. On the way down, it dissipates 1.1 mJ of heat due to air resistance. At what speed, in m/s, does it hit the ground?  
   > Solution, p. 561
7. At a given temperature, the average kinetic energy per molecule is a fixed value, so for instance in air, the more massive oxygen molecules are moving more slowly on the average than the nitrogen molecules. The ratio of the masses of oxygen and nitrogen molecules is 16.00 to 14.01. Now suppose a vessel containing some air is surrounded by a vacuum, and the vessel has a tiny hole in it, which allows the air to slowly leak out. The molecules are bouncing around randomly, so a given molecule will have to “try” many times before it gets lucky enough to head out through the hole. Find the rate at which oxygen leaks divided by the rate at which nitrogen leaks. (Define this rate according to the fraction of the gas that leaks out in a given time, not the mass or number of molecules leaked per unit time.)

8. A person on a bicycle is to coast down a ramp of height $h$ and then pass through a circular loop of radius $r$. What is the smallest value of $h$ for which the cyclist will complete the loop without falling? (Ignore the kinetic energy of the spinning wheels.)

9. Problem 9 has been deleted.

10. Students are often tempted to think of potential energy and kinetic energy as if they were always related to each other, like yin and yang. To show this is incorrect, give examples of physical situations in which (a) PE is converted to another form of PE, and (b) KE is converted to another form of KE. □ Solution, p. 561

11. Lord Kelvin, a physicist, told the story of how he encountered James Joule when Joule was on his honeymoon. As he traveled, Joule would stop with his wife at various waterfalls, and measure the difference in temperature between the top of the waterfall and the still water at the bottom. (a) It would surprise most people to learn that the temperature increased. Why should there be any such effect, and why would Joule care? How would this relate to the energy concept, of which he was the principal inventor? (b) How much of a gain in temperature should there be between the top and bottom of a 50-meter waterfall? (c) What assumptions did you have to make in order to calculate your answer to part b? In reality, would the temperature change be more than or less than what you calculated? [Based on a problem by Arnold Arons.] □

12. Make an order-of-magnitude estimate of the power represented by the loss of gravitational energy of the water going over Niagara Falls. If the hydroelectric plant at the bottom of the falls could convert 100% of this to electrical power, roughly how many households could be powered? □ Solution, p. 561
13 When you buy a helium-filled balloon, the seller has to inflate it from a large metal cylinder of the compressed gas. The helium inside the cylinder has energy, as can be demonstrated for example by releasing a little of it into the air: you hear a hissing sound, and that sound energy must have come from somewhere. The total amount of energy in the cylinder is very large, and if the valve is inadvertently damaged or broken off, the cylinder can behave like a bomb or a rocket.

Suppose the company that puts the gas in the cylinders prepares cylinder A with half the normal amount of pure helium, and cylinder B with the normal amount. Cylinder B has twice as much energy, and yet the temperatures of both cylinders are the same. Explain, at the atomic level, what form of energy is involved, and why cylinder B has twice as much.

14 Explain in terms of conservation of energy why sweating cools your body, even though the sweat is at the same temperature as your body. Describe the forms of energy involved in this energy transformation. Why don’t you get the same cooling effect if you wipe the sweat off with a towel? Hint: The sweat is evaporating.

15 (a) A circular hoop of mass \( m \) and radius \( r \) spins like a wheel while its center remains at rest. Let \( \omega \) (Greek letter omega) be the number of radians it covers per unit time, i.e., \( \omega = \frac{2\pi}{T} \), where the period, \( T \), is the time for one revolution. Show that its kinetic energy equals \( \frac{1}{2}m\omega^2r^2 \).

(b) Show that the answer to part a has the right units. (Note that radians aren’t really units, since the definition of a radian is a unitless ratio of two lengths.)

(c) If such a hoop rolls with its center moving at velocity \( v \), its kinetic energy equals \( \frac{1}{2}mv^2 \), plus the amount of kinetic energy found in part a. Show that a hoop rolls down an inclined plane with half the acceleration that a frictionless sliding block would have.

16 A skateboarder starts at rest nearly at the top of a giant cylinder, and begins rolling down its side. (If he started exactly at rest and exactly at the top, he would never get going!) Show that his board loses contact with the pipe after he has dropped by a height equal to one third the radius of the pipe. > Solution, p. 561

17 In example 10 on page 86, I remarked that accelerating a macroscopic (i.e., not microscopic) object to close to the speed of light would require an unreasonable amount of energy. Suppose that the starship Enterprise from Star Trek has a mass of \( 8.0 \times 10^7 \) kg, about the same as the Queen Elizabeth II. Compute the kinetic energy it would have to have if it was moving at half the speed of light. Compare with the total energy content of the world’s nuclear arsenals, which is about \( 10^{21} \) J.
18  (a) A free neutron (as opposed to a neutron bound into an atomic nucleus) is unstable, and undergoes spontaneous radioactive decay into a proton, an electron, and an antineutrino. The masses of the particles involved are as follows:

- neutron: $1.67495 \times 10^{-27}$ kg
- proton: $1.67265 \times 10^{-27}$ kg
- electron: $0.00091 \times 10^{-27}$ kg
- antineutrino: $< 10^{-35}$ kg

Find the energy released in the decay of a free neutron. √

(b) Neutrons and protons make up essentially all of the mass of the ordinary matter around us. We observe that the universe around us has no free neutrons, but lots of free protons (the nuclei of hydrogen, which is the element that 90% of the universe is made of). We find neutrons only inside nuclei along with other neutrons and protons, not on their own.

If there are processes that can convert neutrons into protons, we might imagine that there could also be proton-to-neutron conversions, and indeed such a process does occur sometimes in nuclei that contain both neutrons and protons: a proton can decay into a neutron, a positron, and a neutrino. A positron is a particle with the same properties as an electron, except that its electrical charge is positive. A neutrino, like an antineutrino, has negligible mass.

Although such a process can occur within a nucleus, explain why it cannot happen to a free proton. (If it could, hydrogen would be radioactive, and you wouldn’t exist!)

19  A little kid in my neighborhood came home from shopping with his mother. They live on a hill, with their driveway oriented perpendicular to the slope. Their minivan was parked in the driveway, and while she was bringing groceries inside, he unlocked the parking brake and put the car in neutral. The steering wheel was locked with the wheels banked. The car rolled downhill in a circular arc with the driveway at its top, eventually crashing through the wall of a neighbor’s living room. (Nobody was hurt.) Suppose the neighbor’s house hadn’t intervened. The car just rolls freely, and we want to know whether it will ever skid. Static friction acts between the asphalt and the tires with coefficient $\mu_s$, the radius of the circle is $r$, the slope of the hill is $\theta$, and the gravitational field has strength $g$. Find the maximum value of $\theta$ such that the car will never skid. √
20  The figure shows two unequal masses, $M$ and $m$, connected by a string running over a pulley. This system was analyzed previously in problem 20 on p. 196, using Newton’s laws.

(a) Analyze the system using conservation of energy instead. Find the speed the weights gain after being released from rest and traveling a distance $h$.  
(b) Use your result from part a to find the acceleration, reproducing the result of the earlier problem.

21  In 2003, physicist and philosopher John Norton came up with the following apparent paradox, in which Newton’s laws, which appear deterministic, can produce nondeterministic results. Suppose that a bead moves frictionlessly on a curved wire under the influence of gravity. The shape of the wire is defined by the function $y(x)$, which passes through the origin, and the bead is released from rest at the origin. For convenience of notation, choose units such that $g = 1$, and define $\dot{y} = \frac{dy}{dt}$ and $y' = \frac{dy}{dx}$.

(a) Show that the equation of motion is

$$y = -\frac{1}{2} \dot{y}^2 \left(1 + y'^{-2}\right).$$

(b) To simplify the calculations, assume from now on that $y' \ll 1$. Find a shape for the wire such that $x = t^4$ is a solution. (Ignore units.)

(c) Show that not just the motion assumed in part b, but any motion of the following form is a solution:

$$x = \begin{cases} 
0 & \text{if } t \leq t_0 \\
(t-t_0)^4 & \text{if } t \geq t_0
\end{cases}$$

This is remarkable because there is no physical principle that determines $t_0$, so if we place the bead at rest at the origin, there is no way to predict when it will start moving.

22  The rock climber in the figure has mass $m$ and is on a slope $\theta$ above the horizontal. At a distance $x$ down the slope below him is a ledge. He is tied in to a climbing rope and being belayed from above, so that if he slips he won’t simply plunge to his death. Climbing ropes are intentionally made out of stretchy material so that in a fall, the climber gets a gentle catch rather than a violent force that would hurt (see example 2, p. 364). However, the rope should not be more stretchy than necessary because of situations like this one: if the rope were to stretch by more than $x$, the climber would hit the ledge.

(a) Find the spring constant that the rope should have in order to limit the amount of rope stretch to $x$.

(b) Show that your answer to part a has the right units.

(c) Analyze the mathematical dependence of the result on each of the variables, and verify that it makes sense physically.
Chapter 13
Work: The Transfer of Mechanical Energy

13.1 Work: the transfer of mechanical energy

The concept of work

The mass contained in a closed system is a conserved quantity, but if the system is not closed, we also have ways of measuring the amount of mass that goes in or out. The water company does this with a meter that records your water use.

Likewise, we often have a system that is not closed, and would like to know how much energy comes in or out. Energy, however, is not a physical substance like water, so energy transfer cannot be measured with the same kind of meter. How can we tell, for instance, how much useful energy a tractor can “put out” on one tank of gas?

The law of conservation of energy guarantees that all the chem-
Work is a transfer of energy.

The tractor raises the weight over the pulley, increasing its gravitational potential energy.

The tractor accelerates the trailer, increasing its kinetic energy.

The tractor pulls a plow. Energy is expended in frictional heating of the plow and the dirt, and in breaking dirt clods and lifting dirt up to the sides of the furrow.

c / The tractor accelerates the trailer, increasing its kinetic energy.

d / The tractor pulls a plow. Energy is expended in frictional heating of the plow and the dirt, and in breaking dirt clods and lifting dirt up to the sides of the furrow.

The conduction of heat is to be distinguished from heating by friction. When a hot potato heats up your hands by conduction, the energy transfer occurs without any force, but when friction heats your car’s brake shoes, there is a force involved. The transfer of energy with and without a force are measured by completely different methods, so we wish to include heat transfer by frictional heating under the definition of work, but not heat transfer by conduction. The definition of work could thus be restated as the amount of energy transferred by forces.

Calculating work as force multiplied by distance

The examples in figures b-d show that there are many different ways in which energy can be transferred. Even so, all these examples have two things in common:

1. A force is involved.
2. The tractor travels some distance as it does the work.

In b, the increase in the height of the weight, \( \Delta y \), is the same as the distance the tractor travels, which we’ll call \( d \). For simplicity, we discuss the case where the tractor raises the weight at constant speed, so that there is no change in the kinetic energy of the weight, and we assume that there is negligible friction in the pulley, so that the force the tractor applies to the rope is the same as the rope’s upward force on the weight. By Newton’s first law, these forces are also of the same magnitude as the earth’s gravitational force on the weight. The increase in the weight’s potential energy is given by \( F \Delta y \), so the work done by the tractor on the weight equals \( Fd \), the product of the force and the distance moved:

\[
W = Fd.
\]
In example c, the tractor’s force on the trailer accelerates it, increasing its kinetic energy. If frictional forces on the trailer are negligible, then the increase in the trailer’s kinetic energy can be found using the same algebra that was used on page 343 to find the potential energy due to gravity. Just as in example b, we have

\[ W = Fd. \]

Does this equation always give the right answer? Well, sort of. In example d, there are two quantities of work you might want to calculate: the work done by the tractor on the plow and the work done by the plow on the dirt. These two quantities can’t both equal \( Fd \). Most of the energy transmitted through the cable goes into frictional heating of the plow and the dirt. The work done by the plow on the dirt is less than the work done by the tractor on the plow, by an amount equal to the heat absorbed by the plow. It turns out that the equation \( W = Fd \) gives the work done by the tractor, not the work done by the plow. How are you supposed to know when the equation will work and when it won’t? The somewhat complex answer is postponed until section 13.6. Until then, we will restrict ourselves to examples in which \( W = Fd \) gives the right answer; essentially the reason the ambiguities come up is that when one surface is slipping past another, \( d \) may be hard to define, because the two surfaces move different distances.

We have also been using examples in which the force is in the same direction as the motion, and the force is constant. (If the force was not constant, we would have to represent it with a function, not a symbol that stands for a number.) To summarize, we have:

---

**rule for calculating work (simplest version)**

The work done by a force can be calculated as

\[ W = Fd, \]

if the force is constant and in the same direction as the motion. Some ambiguities are encountered in cases such as kinetic friction.
Example 1.

Example 2. Surprisingly, the climber is in more danger at 1 than at 2. The distance \( d \) is the amount by which the rope will stretch while work is done to transfer the kinetic energy of a fall out of her body.

Mechanical work done in an earthquake example 1

In 1998, geologists discovered evidence for a big prehistoric earthquake in Pasadena, between 10,000 and 15,000 years ago. They found that the two sides of the fault moved 6.7 m relative to one another, and estimated that the force between them was \( 1.3 \times 10^{17} \) N. How much energy was released?

Multiplying the force by the distance gives \( 9 \times 10^{17} \) J. For comparison, the Northridge earthquake of 1994, which killed 57 people and did 40 billion dollars of damage, released 22 times less energy.

The fall factor example 2

Counterintuitively, the rock climber may be in more danger in figure g/1 than later when she gets up to position g/2.

Along her route, the climber has placed removable rock anchors (not shown) and carabiners attached to the anchors. She clips the rope into each carabiner so that it can travel but can't pop out. In both 1 and 2, she has ascended a certain distance above her last anchor, so that if she falls, she will drop through a height \( h \) that is about twice this distance, and this fall height is about the same in both cases. In fact, \( h \) is somewhat larger than twice her height above the last anchor, because the rope is intentionally designed to stretch under the big force of a falling climber who suddenly brings it taut.

To see why we want a stretchy rope, consider the equation \( F = \frac{W}{d} \) in the case where \( d \) is zero; \( F \) would theoretically become infinite. In a fall, the climber loses a fixed amount of gravitational energy \( mgh \). This is transformed into an equal amount of kinetic energy as she falls, and eventually this kinetic energy has to be transferred out of her body when the rope comes up taut. If the rope was not stretchy, then the distance traveled at the point where the rope attaches to her harness would be zero, and the force exerted would theoretically be infinite. Before the rope reached the theoretically infinite tension \( F \) it would break (or her back would break, or her anchors would be pulled out of the rock). We want the rope to be stretchy enough to make \( d \) fairly big, so that dividing \( W \) by \( d \) gives a small force.  

In g/1 and g/2, the fall \( h \) is about the same. What is different is the length \( L \) of rope that has been paid out. A longer rope can stretch more, so the distance \( d \) traveled after the “catch” is proportional to \( L \). Combining \( F = \frac{W}{d}, W \propto h, \) and \( d \propto L \), we have \( F \propto h/L \). For these reasons, rock climbers define a fall factor \( f = h/L \). The larger fall factor in g/1 is more dangerous.

---

1 Actually \( F \) isn’t constant, because the tension in the rope increases steadily as it stretches, but this is irrelevant to the present analysis.
Machines can increase force, but not work.

Figure h shows a pulley arrangement for doubling the force supplied by the tractor (book 1, section 5.6). The tension in the left-hand rope is equal throughout, assuming negligible friction, so there are two forces pulling the pulley to the left, each equal to the original force exerted by the tractor on the rope. This doubled force is transmitted through the right-hand rope to the stump.

It might seem as though this arrangement would also double the work done by the tractor, but look again. As the tractor moves forward 2 meters, 1 meter of rope comes around the pulley, and the pulley moves 1 m to the left. Although the pulley exerts double the force on the stump, the pulley and stump only move half as far, so the work done on the stump is no greater that it would have been without the pulley.

The same is true for any mechanical arrangement that increases or decreases force, such as the gears on a ten-speed bike. You can’t get out more work than you put in, because that would violate conservation of energy. If you shift gears so that your force on the pedals is amplified, the result is that you just have to spin the pedals more times.

No work is done without motion.

It strikes most students as nonsensical when they are told that if they stand still and hold a heavy bag of cement, they are doing no work on the bag. Even if it makes sense mathematically that $W = Fd$ gives zero when $d$ is zero, it seems to violate common sense. You would certainly become tired! The solution is simple. Physicists have taken over the common word “work” and given it a new technical meaning, which is the transfer of energy. The energy of the bag of cement is not changing, and that is what the physicist means by saying no work is done on the bag.

There is a transformation of energy, but it is taking place entirely within your own muscles, which are converting chemical energy into heat. Physiologically, a human muscle is not like a tree limb, which can support a weight indefinitely without the expenditure of energy. Each muscle cell’s contraction is generated by zillions of little molecular machines, which take turns supporting the tension. When a
Whenever energy is transferred out of the spring, the same amount has to be transferred into the ball, and vice versa. As the spring compresses, the ball is doing positive work on the spring (giving up its KE and transferring energy into the spring as PE), and as it decompresses the ball is doing negative work (extracting energy).

Positive and negative work

When object A transfers energy to object B, we say that A does positive work on B. B is said to do negative work on A. In other words, a machine like a tractor is defined as doing positive work. This use of the plus and minus signs relates in a logical and consistent way to their use in indicating the directions of force and motion in one dimension. In figure i, suppose we choose a coordinate system with the x axis pointing to the right. Then the force the spring exerts on the ball is always a positive number. The ball’s motion, however, changes directions. The symbol $d$ is really just a shorter way of writing the familiar quantity $\Delta x$, whose positive and negative signs indicate direction.

While the ball is moving to the left, we use $d < 0$ to represent its direction of motion, and the work done by the spring, $Fd$, comes out negative. This indicates that the spring is taking kinetic energy out of the ball, and accepting it in the form of its own potential energy.

As the ball is reaccelerated to the right, it has $d > 0$, $Fd$ is positive, and the spring does positive work on the ball. Potential energy is transferred out of the spring and deposited in the ball as kinetic energy.

In summary:

rule for calculating work (including cases of negative work)

The work done by a force can be calculated as

$$W = Fd,$$

if the force is constant and along the same line as the motion. The quantity $d$ is to be interpreted as a synonym for $\Delta x$, i.e., positive and negative signs are used to indicate the direction of motion. Some ambiguities are encountered in cases such as kinetic friction.

self-check B

In figure i, what about the work done by the ball on the spring?

Answer, p. 568

There are many examples where the transfer of energy out of an object cancels out the transfer of energy in. When the tractor pulls the plow with a rope, the rope does negative work on the tractor and positive work on the plow. The total work done by the rope is zero, which makes sense, since it is not changing its energy.

It may seem that when your arms do negative work by lowering
a bag of cement, the cement is not really transferring energy into your body. If your body was storing potential energy like a compressed spring, you would be able to raise and lower a weight all day, recycling the same energy. The bag of cement does transfer energy into your body, but your body accepts it as heat, not as potential energy. The tension in the muscles that control the speed of the motion also results in the conversion of chemical energy to heat, for the same physiological reasons discussed previously in the case where you just hold the bag still.

One of the advantages of electric cars over gasoline-powered cars is that it is just as easy to put energy back in a battery as it is to take energy out. When you step on the brakes in a gas car, the brake shoes do negative work on the rest of the car. The kinetic energy of the car is transmitted through the brakes and accepted by the brake shoes in the form of heat. The energy cannot be recovered. Electric cars, however, are designed to use regenerative braking. The brakes don’t use friction at all. They are electrical, and when you step on the brake, the negative work done by the brakes means they accept the energy and put it in the battery for later use. This is one of the reasons why an electric car is far better for the environment than a gas car, even if the ultimate source of the electrical energy happens to be the burning of oil in the electric company’s plant. The electric car recycles the same energy over and over, and only dissipates heat due to air friction and rolling resistance, not braking. (The electric company’s power plant can also be fitted with expensive pollution-reduction equipment that would be prohibitively expensive or bulky for a passenger car.)
Discussion questions

A Besides the presence of a force, what other things differentiate the processes of frictional heating and heat conduction?

B Criticize the following incorrect statement: “A force doesn’t do any work unless it’s causing the object to move.”

C To stop your car, you must first have time to react, and then it takes some time for the car to slow down. Both of these times contribute to the distance you will travel before you can stop. The figure shows how the average stopping distance increases with speed. Because the stopping distance increases more and more rapidly as you go faster, the rule of one car length per 10 m.p.h. of speed is not conservative enough at high speeds. In terms of work and kinetic energy, what is the reason for the more rapid increase at high speeds?

Discussion question C.

13.2 Work in three dimensions

A force perpendicular to the motion does no work.

Suppose work is being done to change an object’s kinetic energy. A force in the same direction as its motion will speed it up, and a force in the opposite direction will slow it down. As we have already seen, this is described as doing positive work or doing negative work on the object. All the examples discussed up until now have been of motion in one dimension, but in three dimensions the force can be at any angle \( \theta \) with respect to the direction of motion.

What if the force is perpendicular to the direction of motion? We have already seen that a force perpendicular to the motion results in circular motion at constant speed. The kinetic energy does not change, and we conclude that no work is done when the force is perpendicular to the motion.

So far we have been reasoning about the case of a single force acting on an object, and changing only its kinetic energy. The result is more generally true, however. For instance, imagine a hockey puck sliding across the ice. The ice makes an upward normal force, but does not transfer energy to or from the puck.
**Forces at other angles**

Suppose the force is at some other angle with respect to the motion, say \( \theta = 45^\circ \). Such a force could be broken down into two components, one along the direction of the motion and the other perpendicular to it. The force vector equals the vector sum of its two components, and the principle of vector addition of forces thus tells us that the work done by the total force cannot be any different than the sum of the works that would be done by the two forces by themselves. Since the component perpendicular to the motion does no work, the work done by the force must be

\[
W = F_\parallel |\mathbf{d}|, \quad \text{[work done by a constant force]}
\]

where the vector \( \mathbf{d} \) is simply a less cumbersome version of the notation \( \Delta \mathbf{r} \). This result can be rewritten via trigonometry as

\[
W = |\mathbf{F}| |\mathbf{d}| \cos \theta. \quad \text{[work done by a constant force]}
\]

Even though this equation has vectors in it, it depends only on their magnitudes, and the magnitude of a vector is a scalar. Work is therefore still a scalar quantity, which only makes sense if it is defined as the transfer of energy. Ten gallons of gasoline have the ability to do a certain amount of mechanical work, and when you pull in to a full-service gas station you don’t have to say “Fill ’er up with 10 gallons of south-going gas.”

Students often wonder why this equation involves a cosine rather than a sine, or ask if it would ever be a sine. In vector addition, the treatment of sines and cosines seemed more equal and democratic, so why is the cosine so special now? The answer is that if we are going to describe, say, a velocity vector, we must give both the component parallel to the \( x \) axis and the component perpendicular to the \( x \) axis (i.e., the \( y \) component). In calculating work, however, the force component perpendicular to the motion is irrelevant — it changes the direction of motion without increasing or decreasing the energy of the object on which it acts. In this context, it is only the parallel force component that matters, so only the cosine occurs.

**self-check C**

(a) Work is the transfer of energy. According to this definition, is the horse in the picture doing work on the pack? (b) If you calculate work by the method described in this section, is the horse in figure o doing work on the pack?  

\( \triangleright \) Answer, p. 568

\(^1\)Pushing a broom  example 3

\( \triangleright \) If you exert a force of 21 N on a push broom, at an angle 35 degrees below horizontal, and walk for 5.0 m, how much work do you do? What is the physical significance of this quantity of work?

\( \triangleright \) Using the second equation above, the work done equals

\[
(21 \text{ N})(5.0 \text{ m})(\cos 35^\circ) = 86 \text{ J}.
\]
The form of energy being transferred is heat in the floor and the broom's bristles. This comes from the chemical energy stored in your body. (The majority of the calories you burn are dissipated directly as heat inside your body rather than doing any work on the broom. The 86 J is only the amount of energy transferred through the broom's handle.)

A violin example 4

As a violinist draws the bow across a string, the bow hairs exert both a normal force and a kinetic frictional force on the string. The normal force is perpendicular to the direction of motion, and does no work. However, the frictional force is in the same direction as the motion of the bow, so it does work: energy is transferred to the string, causing it to vibrate.

One way of playing a violin more loudly is to use longer strokes. Since \( W = Fd \), the greater distance results in more work.

A second way of getting a louder sound is to press the bow more firmly against the strings. This increases the normal force, and although the normal force itself does no work, an increase in the normal force has the side effect of increasing the frictional force, thereby increasing \( W = Fd \).

The violinist moves the bow back and forth, and sound is produced on both the “up-bow” (the stroke toward the player's left) and the “down-bow” (to the right). One may, for example, play a series of notes in alternation between up-bows and down-bows. However, if the notes are of unequal length, the up and down motions tend to be unequal, and if the player is not careful, she can run out of bow in the middle of a note! To keep this from happening, one can move the bow more quickly on the shorter notes, but the resulting increase in \( d \) will make the shorter notes louder than they should be. A skilled player compensates by reducing the force.

13.3 The dot product

Up until now, we have not found any physically useful way to define the multiplication of two vectors. It would be possible, for instance, to multiply two vectors component by component to form a third vector, but there are no physical situations where such a multiplication would be useful.

The equation \( W = |\mathbf{F}||\mathbf{d}| \cos \theta \) is an example of a sort of multiplication of vectors that is useful. The result is a scalar, not a vector, and this is therefore often referred to as the scalar product of the vectors \( \mathbf{F} \) and \( \mathbf{d} \). There is a standard shorthand notation for
this operation,

\[ \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta, \]

[definition of the notation \( \mathbf{A} \cdot \mathbf{B} \);
\( \theta \) is the angle between vectors \( \mathbf{A} \) and \( \mathbf{B} \)]

and because of this notation, a more common term for this operation
is the *dot product*. In dot product notation, the equation for work
is simply

\[ W = \mathbf{F} \cdot \mathbf{d}. \]

The dot product has the following geometric interpretation:

\[ \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|(\text{component of } \mathbf{B} \text{ parallel to } \mathbf{A}) \]
\[ = |\mathbf{B}|(\text{component of } \mathbf{A} \text{ parallel to } \mathbf{B}) \]

The dot product has some of the properties possessed by ordinary
multiplication of numbers,

\[ \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \]
\[ \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \]
\[ (c\mathbf{A}) \cdot \mathbf{B} = c(\mathbf{A} \cdot \mathbf{B}), \]

but it lacks one other: the ability to undo multiplication by dividing.

If you know the components of two vectors, you can easily cal-
culate their dot product as follows:

\[ \mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \]

(This can be proved by first analyzing the special case where each
vector has only an \( x \) component, and the similar cases for \( y \) and \( z \).
We can then use the rule \( \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \)
to make a generalization by writing each vector as the sum of its \( x \), \( y \), and \( z \)
components. See homework problem 23.)

1*Magnitude expressed with a dot product*  
*example 5*

If we take the dot product of any vector \( \mathbf{b} \) with itself, we find

\[ \mathbf{b} \cdot \mathbf{b} = (b_x\hat{x} + b_y\hat{y} + b_z\hat{z}) \cdot (b_x\hat{x} + b_y\hat{y} + b_z\hat{z}) \]
\[ = b_x^2 + b_y^2 + b_z^2, \]

so its magnitude can be expressed as

\[ |\mathbf{b}| = \sqrt{\mathbf{b} \cdot \mathbf{b}}. \]

We will often write \( b^2 \) to mean \( \mathbf{b} \cdot \mathbf{b} \), when the context makes
it clear what is intended. For example, we could express kinetic
energy as \((1/2)m|\mathbf{v}|^2\), \((1/2)m\mathbf{v} \cdot \mathbf{v}\), or \((1/2)m\mathbf{v}^2\). In the third version,
nothing but context tells us that \( \mathbf{v} \) really stands for the magnitude
of some vector \( \mathbf{v} \).
13.4 Varying force

Up until now we have done no actual calculations of work in cases where the force was not constant. The question of how to treat such cases is mathematically analogous to the issue of how to generalize the equation (distance) = (velocity)(time) to cases where the velocity was not constant. We have to make the equation into an integral:

\[ W = \int F \, dx \]

The examples in this section are ones in which the force is varying, but is always along the same line as the motion.

**self-check D**
In which of the following examples would it be OK to calculate work using \( F \, d \), and in which ones would you have to integrate?

(a) A fishing boat cruises with a net dragging behind it.
(b) A magnet leaps onto a refrigerator from a distance.
(c) Earth’s gravity does work on an outward-bound space probe.

Answer, p. 568

**Work done by a spring example 7**

An important and straightforward example is the calculation of the work done by a spring that obeys Hooke’s law,

\[ F \approx -k(x - x_0), \]

where \( x_0 \) is the equilibrium position and the minus sign is because this is the force being exerted by the spring, not the force that would have to act on the spring to keep it at this position. That is, if the position of the cart in figure p is to the right of equilibrium, the spring pulls back to the left, and vice-versa. Integrating, we find that the work done between \( x_1 \) and \( x_2 \) is

\[ W = -\frac{1}{2}k(x - x_0)^2 \bigg|_{x_1}^{x_2}. \]
which can be integrated to give

\[ W = \int_{r_1}^{r_2} \frac{GMm}{r^2} \, dr \]

\[ = -GMm \left( \frac{1}{r_2} - \frac{1}{r_1} \right). \]

### 13.5 Work and potential energy

The techniques for calculating work can also be applied to the calculation of potential energy. If a certain force depends only on the distance between the two participating objects, then the energy released by changing the distance between them is defined as the potential energy, and the amount of potential energy lost equals minus the work done by the force,

\[ \Delta PE = -W. \]

The minus sign occurs because positive work indicates that the potential energy is being expended and converted to some other form.

It is sometimes convenient to pick some arbitrary position as a reference position, and derive an equation for once and for all that gives the potential energy relative to this position

\[ PE_x = -W_{\text{ref} \rightarrow x}. \] [potential energy at a point \( x \)]

To find the energy transferred into or out of potential energy, one then subtracts two different values of this equation.

These equations might almost make it look as though work and energy were the same thing, but they are not. First, potential energy measures the energy that a system has stored in it, while work measures how much energy is transferred in or out. Second, the techniques for calculating work can be used to find the amount of energy transferred in many situations where there is no potential energy involved, as when we calculate the amount of kinetic energy transformed into heat by a car’s brake shoes.

**Example 9**

A toy gun uses a spring with a spring constant of 10 N/m to shoot a ping-pong ball of mass 5 g. The spring is compressed to 10 cm shorter than its equilibrium length when the gun is loaded. At what speed is the ball released?

The equilibrium point is the natural choice for a reference point. Using the equation found previously for the work, we have

\[ PE_x = \frac{1}{2} k (x - x_0)^2. \]
The spring loses contact with the ball at the equilibrium point, so
the final potential energy is

$$PE_f = 0.$$  

The initial potential energy is

$$PE_i = \frac{1}{2}(10 \text{ N/m})(0.10 \text{ m})^2.$$  

$$= 0.05 \text{ J.}$$  

The loss in potential energy of 0.05 J means an increase in kinetic
energy of the same amount. The velocity of the ball is found by
solving the equation $KE = (1/2)mv^2$ for $v$,

$$v = \sqrt{\frac{2KE}{m}}$$  

$$= \sqrt{\frac{(2)(0.05 \text{ J})}{0.005 \text{ kg}}}$$  

$$= 4 \text{ m/s.}$$

Gravitational potential energy example 10

We have already found the equation $\Delta PE = -F\Delta y$ for the
gravitational potential energy when the change in height is not
enough to cause a significant change in the gravitational force $F$.
What if the change in height is enough so that this assumption
is no longer valid? Use the equation $W = GMm(1/r_2 - 1/r_1)$
derived in example 8 to find the potential energy, using $r = \infty$ as
a reference point.

The potential energy equals minus the work that would have to
be done to bring the object from $r_1 = \infty$ to $r = r_2$, which is

$$PE = -\frac{GMm}{r}.$$  

This is simpler than the equation for the work, which is an exam-
ple of why it is advantageous to record an equation for potential
energy relative to some reference point, rather than an equation
for work.

Although the equations derived in the previous two examples
may seem arcane and not particularly useful except for toy design-
ers and rocket scientists, their usefulness is actually greater than
it appears. The equation for the potential energy of a spring can
be adapted to any other case in which an object is compressed,
stretched, twisted, or bent. While you are not likely to use the
equation for gravitational potential energy for anything practical, it
is directly analogous to an equation that is extremely useful in chem-
istry, which is the equation for the potential energy of an electron.
at a distance \( r \) from the nucleus of its atom. As discussed in more
detail later in the course, the electrical force between the electron
and the nucleus is proportional to \( 1/r^2 \), just like the gravitational
force between two masses. Since the equation for the force is of the
same form, so is the equation for the potential energy.

Discussion questions

A. What does the graph of \( PE = (1/2)k(x - x_o)^2 \) look like as a function
   of \( x \)? Discuss the physical significance of its features.

B. What does the graph of \( PE = -GMm/r \) look like as a function of \( r \)?
   Discuss the physical significance of its features. How would the equation
   and graph change if some other reference point was chosen rather than
   \( r = \infty \)?

C. Starting at a distance \( r \) from a planet of mass \( M \), how fast must an
   object be moving in order to have a hyperbolic orbit, i.e., one that never
   comes back to the planet? This velocity is called the escape velocity. Inter-
   preting the result, does it matter in what direction the velocity is? Does
   it matter what mass the object has? Does the object escape because it is
   moving too fast for gravity to act on it?

D. Does a spring have an “escape velocity?”

E. If the form of energy being transferred is potential energy, then the
   equations \( F = dW/dx \) and \( W = \int Fdx \) become \( F = -dPE/dx \) and
   \( PE = -\int Fdx \). How would you then apply the following calculus con-
   cepts: zero derivative at minima and maxima, and the second derivative
   test for concavity up or down.
The work-KE theorem

Proof

For simplicity, we have assumed $F_{\text{total}}$ to be constant, and therefore $a_{cm} = F_{\text{total}} / m$ is also constant, and the constant-acceleration equation

$$v_{cm,f}^2 = v_{cm,i}^2 + 2a_{cm}\Delta x_{cm}$$

applies. Multiplying by $m/2$ on both sides and applying Newton's second law gives

$$K_{E_{cm,f}}^2 = K_{E_{cm,i}}^2 + F_{\text{total}}\Delta x_{cm},$$

which is the result that was to be proved.

Further interpretation

The logical structure of this book is that although Newton's laws are discussed before conservation laws, the conservation laws are taken to be fundamental, since they are true even in cases where Newton's laws fail. Many treatments of this subject present the work-KE theorem as a proof that kinetic energy behaves as $(1/2)mv^2$. This is a matter of taste, but one can just as well rearrange the equations in the proof above to solve for the unknown $a_{cm}$ and prove Newton's second law as a consequence of conservation of energy. Ultimately we have a great deal of freedom in choosing which equations to take as definitions, which to take as empirically verified laws of nature, and which to take as theorems.

Regardless of how we slice things, we require both mathematical consistency and consistency with experiment. As described on p. 323, the work-KE theorem is an important part of this interlocking system of relationships.

13.6 When does work equal force times distance?

In the example of the tractor pulling the plow discussed on page 363, the work did not equal $Fd$. The purpose of this section is to explain more fully how the quantity $Fd$ can and cannot be used. To simplify things, I write $Fd$ throughout this section, but more generally everything said here would be true for the area under the graph of $F_\parallel$ versus $d$.

The following two theorems allow most of the ambiguity to be cleared up.

The work-kinetic-energy theorem

The change in kinetic energy associated with the motion of an object’s center of mass is related to the total force acting on it and to the distance traveled by its center of mass according to the equation $\Delta KE_{cm} = F_{\text{total}}d_{cm}$.

A proof is given in the sidebar, along with some interpretation of how this result relates to the logical structure of our presentation. Note that despite the traditional name, it does not necessarily tell the amount of work done, since the forces acting on the object could be changing other types of energy besides the KE associated with its center of mass motion.

The second theorem does relate directly to work:

When a contact force acts between two objects and the two surfaces do not slip past each other, the work done equals $F d$, where $d$ is the distance traveled by the point of contact.

This one has no generally accepted name, so we refer to it simply as the second theorem.

A great number of physical situations can be analyzed with these two theorems, and often it is advantageous to apply both of them to the same situation.

An ice skater pushing off from a wall example 11

The work-kinetic energy theorem tells us how to calculate the skater’s kinetic energy if we know the amount of force and the distance her center of mass travels while she is pushing off.

The second theorem tells us that the wall does no work on the skater. This makes sense, since the wall does not have any source of energy.

Absorbing an impact without recoiling? example 12

Is it possible to absorb an impact without recoiling? For instance, would a brick wall “give” at all if hit by a ping-pong ball?

There will always be a recoil. In the example proposed, the wall will surely have some energy transferred to it in the form of heat.
and vibration. The second theorem tells us that we can only have nonzero work if the distance traveled by the point of contact is nonzero.

Dragging a refrigerator at constant velocity example 13

Newton’s first law tells us that the total force on the refrigerator must be zero: your force is canceling the floor’s kinetic frictional force. The work-kinetic energy theorem is therefore true but useless. It tells us that there is zero total force on the refrigerator, and that the refrigerator’s kinetic energy doesn’t change.

The second theorem tells us that the work you do equals your hand’s force on the refrigerator multiplied by the distance traveled. Since we know the floor has no source of energy, the only way for the floor and refrigerator to gain energy is from the work you do. We can thus calculate the total heat dissipated by friction in the refrigerator and the floor.

Note that there is no way to find how much of the heat is dissipated in the floor and how much in the refrigerator.

Accelerating a cart example 14

If you push on a cart and accelerate it, there are two forces acting on the cart: your hand’s force, and the static frictional force of the ground pushing on the wheels in the opposite direction.

Applying the second theorem to your force tells us how to calculate the work you do.

Applying the second theorem to the floor’s force tells us that the floor does no work on the cart. There is no motion at the point of contact, because the atoms in the floor are not moving. (The atoms in the surface of the wheel are also momentarily at rest when they touch the floor.) This makes sense, since the floor does not have any source of energy.

The work-kinetic energy theorem refers to the total force, and because the floor’s backward force cancels part of your force, the total force is less than your force. This tells us that only part of your work goes into the kinetic energy associated with the forward motion of the cart’s center of mass. The rest goes into rotation of the wheels.

13.7 Uniqueness of the dot product

In this section I prove that the vector dot product is unique, in the sense that there is no other possible way to define it that is consistent with rotational invariance and that reduces appropriately to ordinary multiplication in one dimension.

Suppose we want to find some way to multiply two vectors to get a scalar, and we don’t know how this operation should be defined.
Let’s consider what we would get by performing this operation on various combinations of the unit vectors \( \hat{x}, \hat{y}, \) and \( \hat{z} \). Rotational invariance requires that we handle the three coordinate axes in the same way, without giving special treatment to any of them, so we must have \( \hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} \) and \( \hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} \). This is supposed to be a way of generalizing ordinary multiplication, so for consistency with the property \( 1 \times 1 = 1 \) of ordinary numbers, the result of multiplying a magnitude-one vector by itself had better be the scalar 1, so \( \hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1 \). Furthermore, there is no way to satisfy rotational invariance unless we define the mixed products to be zero, \( \hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0 \); for example, a 90-degree rotation of our frame of reference about the \( z \) axis reverses the sign of \( \hat{x} \cdot \hat{y} \), but rotational invariance requires that \( \hat{x} \cdot \hat{y} \) produce the same result either way, and zero is the only number that stays the same when we reverse its sign. Establishing these six products of unit vectors suffices to define the operation in general, since any two vectors that we want to multiply can be broken down into components, e.g., \( (2\hat{x} + 3\hat{z}) \cdot \hat{z} = 2\hat{x} \cdot \hat{z} + 3\hat{z} \cdot \hat{z} = 0 + 3 = 3 \). Thus by requiring rotational invariance and consistency with multiplication of ordinary numbers, we find that there is only one possible way to define a multiplication operation on two vectors that gives a scalar as the result. (There is, however, a different operation, discussed in chapter 15, which multiplies two vectors to give a vector.)

13.8 * A dot product for relativity?

In section 13.7 I showed that the dot product is the only physically sensible way to multiply two vectors to get a scalar. This is essentially because the outcome of experiments shouldn’t depend on which way we rotate the laboratory. Dot products relate to the lengths of vectors and the angles between them, and rotations don’t change lengths or angles.

Let’s consider how this would apply to relativity. Relativity tells us that the length of a measuring rod is not absolute. Rotating the lab won’t change its length, but changing the lab’s state of motion will. The rod’s length is greatest in the frame that is at rest relative to the rod. This suggests that relativity requires some new variation on the dot product: some slightly different way of multiplying two vectors to find a number that doesn’t depend on the frame of reference.

Clock time

We do know of a number that stays the same in all frames of reference. In figure am on p. 83, we proved that the Lorentz transformation doesn’t change the area of a shape in the \( x-t \) plane. We used this only as a stepping stone toward the Lorentz transformation, but it is natural to wonder whether this kind of area has any
physical interest of its own.

The equal-area result is not relativistic, since the proof never appeals to property 5 on page 79. Cases I and II on page 82 also have the equal-area property. We can see this clearly in a Galilean transformation like figure ag on p. 80, where the distortion of the rectangle could be accomplished by cutting it into vertical slices and then displacing the slices upward without changing their areas.

But the area does have a nice interpretation in the relativistic case. Suppose that we have events A (Charles VII is restored to the throne) and B (Joan of Arc is executed). Now imagine that technologically advanced aliens want to be present at both A and B, but in the interim they wish to fly away in their spaceship, be present at some other event P (perhaps a news conference at which they give an update on the events taking place on earth), but get back in time for B. Since nothing can go faster than \( c \) (which we take to equal 1 in appropriate units), P cannot be too far away. The set of all possible events P forms a rectangle, figure s/1, in the \( x - t \) plane that has A and B at opposite corners and whose edges have slopes equal to \( \pm 1 \). We call this type of rectangle a light-rectangle, because its sides could represent the motion of rays of light.

The area of this rectangle will be the same regardless of one’s frame of reference. In particular, we could choose a special frame of reference, panel 2 of the figure, such that A and B occur in the same place. (They do not occur at the same place, for example, in the sun’s frame, because the earth is spinning and going around the sun.) Since the speed \( c \), which equals 1 in our units, is the same in all frames of reference, and the sides of the rectangle had slopes \( \pm 1 \) in frame 1, they must still have slopes \( \pm 1 \) in frame 2. The rectangle becomes a square with its diagonals parallel to the \( x \) and \( t \) axes, and the length of these diagonals equals the time \( \tau \) elapsed on a clock that is at rest in frame 2, i.e., a clock that glides through space at constant velocity from A to B, meeting up with the planet earth at the appointed time. As shown in panel 3 of the figure, the area of the gray regions can be interpreted as half the square of this gliding-clock time.

If events A and B are separated by a distance \( x \) and a time \( t \), then the area of the gray region is \( \tau^2/2 \).
in general $t^2 - x^2$ gives the square of the gliding-clock time. Proof:
Based on units, the expression must have the form $(\ldots)t^2 + (\ldots)tx + (\ldots)x^2$, where each $(\ldots)$ represents a unitless constant. The $tx$ coefficient must be zero by property 2 on p. 79. For consistency with figure s/3, the $t^2$ coefficient must equal 1. Since the area vanishes for $x = t$, the $x^2$ coefficient must equal $-1$.

When $|x|$ is greater than $|t|$, events A and B are so far apart in space and so close together in time that it would be impossible to have a causal relationship between them, since $c = 1$ is the maximum speed of cause and effect. In this situation $t^2 - x^2$ is negative and cannot be interpreted as a clock time, but it can be interpreted as minus the square of the distance between A and B as measured by rulers at rest in a frame in which A and B are simultaneous.

**Four-vectors**

No matter what, $t^2 - x^2$ is the same as measured in all frames of reference. Geometrically, it plays the same role in the $x$-$t$ plane that ruler measurements play in the Euclidean plane. In Euclidean geometry, the ruler-distance between any two points stays the same regardless of rotation, i.e., regardless of the angle from which we view the scene; according to the Pythagorean theorem, the square of this distance is $x^2 + y^2$. In the $x$-$t$ plane, $t^2 - x^2$ stays the same regardless of the frame of reference. This suggests that by analogy with the dot product

$$x_1x_2 + y_1y_2$$

in the Euclidean $x$-$y$ plane, we define a similar operation in the $x$-$t$ plane,

$$t_1t_2 - x_1x_2.$$

Putting in the other two spatial dimensions, we have

$$t_1t_2 - x_1x_2 - y_1y_2 - z_1z_2.$$

A mathematical object like $(t, x, y, z)$ is referred to as a four-vector, as opposed to a three-vector like $(x, y, z)$. The term “dot product” has connotations of referring only to three-vectors, so the operation of taking the scalar product of two four-vectors is usually referred to instead as the “inner product.” There are various ways of notating the inner product of vectors $\mathbf{a}$ and $\mathbf{b}$, such as $\mathbf{a} \cdot \mathbf{b}$ or $\langle \mathbf{a}, \mathbf{b} \rangle$.

The magnitude of a three-vector is defined by taking the square root of its dot product with itself, and this square root is always a real number, because a vector’s dot product with itself is always positive. But the inner product of a four-vector with itself can be positive, zero, or negative, and in these cases the vector is referred to as timelike, lightlike, spacelike, respectively. Since material objects can never go as fast as $c$, the vector $(\Delta t, \Delta x, \Delta y, \Delta z)$ describing an object’s motion from one event to another is always timelike.
One of the classic paradoxes of relativity, known as the twin paradox, is usually stated something like this. Alice and Betty are identical twins. Betty goes on a space voyage at relativistic speeds, traveling away from the earth and then turning around and coming back. Meanwhile, Alice stays on earth. When Betty returns, she is younger than Alice because of relativistic time dilation. But isn't it valid to say that Betty's spaceship is standing still and the earth moving? In that description, wouldn't Alice end up younger and Betty older?

The most common way of explaining the non-paradoxical nature of this paradox is that although special relativity says that inertial motion is relative, it doesn't say that noninertial motion is relative. In this respect it is the same as Newtonian mechanics. Betty experiences accelerations on her voyage, but Alice doesn't. Therefore there is no doubt about who actually went on the trip and who didn't.

This resolution, however, may not be entirely satisfying because it makes it sound as if relativistic time dilation is not occurring while Betty's ship cruises at constant velocity, but only while the ship is speeding up or slowing down. This would appear to contradict our earlier interpretation of relativistic time dilation, which was that a clock runs fastest according to an observer at rest relative to the clock. Furthermore, if it's acceleration that causes the effect, should we be looking for some new formula that computes time dilation based on acceleration?

The first thing to realize is that there is no unambiguous way to decide during which part of Betty's journey the time dilation is occurring. To do this, we could need to be able to compare Alice and Betty's clocks many times over the course of the trip. But neither twin has any way of finding out what her sister's clock reads “now,” except by exchanging radio signals, which travel at the speed of light. The speed-of-light lag vanishes only at the beginning and end of the trip, when the twins are in the same place.

Furthermore, we can use the inner product to show that the accumulated difference in clock time doesn’t depend on the details of how Betty carries out her accelerations and decelerations. In fact, we can get the right answer simply by assuming that these changes in velocities occur instantaneously.

In Euclidean geometry, the triangle inequality $|b + c| < |b| + |c|$ follows from

\[(|b| + |c|)^2 - (b + c) \cdot (b + c) = 2(|b||c| - b \cdot c) \geq 0.\]

The reason this quantity always comes out positive is that for two vectors of fixed magnitude, the greatest dot product is always
achieved in the case where they lie along the same direction.

In the geometry of the $x$-$t$ plane, the situation is different. Suppose that $\mathbf{b}$ and $\mathbf{c}$ are timelike vectors, so that they represent possible $(\Delta t, \Delta x, \ldots)$ vectors for Betty on the outward and return legs of her trip. Then $\mathbf{a} = \mathbf{b} + \mathbf{c}$ describes the vector for Alice’s motion. Alice goes by a direct route through the $x$-$t$ plane while Betty takes a detour. The magnitude of each timelike vector represents the time elapsed on a clock carried by that twin. The triangle equality is now reversed, becoming $|\mathbf{b} + \mathbf{c}| > |\mathbf{b}| + |\mathbf{c}|$. The difference from the Euclidean case arises because inner products are no longer necessarily maximized if vectors are in the same direction. E.g., for two lightlike vectors, $\mathbf{b} \cdot \mathbf{c}$ vanishes entirely if $\mathbf{b}$ and $\mathbf{c}$ are parallel. For timelike vectors, parallelism actually minimizes the inner product rather than maximizing it.\footnote{Proof: Let $\mathbf{b}$ and $\mathbf{c}$ be parallel and timelike, and directed forward in time. Adopt a frame of reference in which every spatial component of each vector vanishes. This entails no loss of generality, since inner products are invariant under such a transformation. Now let $\mathbf{b}$ and $\mathbf{c}$ be pulled away from parallelism, like opening a pair of scissors in the $x$-$t$ plane. This reduces $\mathbf{b} \cdot \mathbf{c}$, while causing $\mathbf{b} \times \mathbf{c}$ to become negative. Both effects increase the inner product.}
Summary

Selected vocabulary
work . . . . . . . . the amount of energy transferred into or out of a system, excluding energy transferred by heat conduction

Notation
$W$ . . . . . . . . work

Summary

Work is a measure of the transfer of mechanical energy, i.e., the transfer of energy by a force rather than by heat conduction. When the force is constant, work can usually be calculated as

$$W = F_\parallel |\mathbf{d}|,$$
[only if the force is constant]

where $\mathbf{d}$ is simply a less cumbersome notation for $\Delta \mathbf{r}$, the vector from the initial position to the final position. Thus,

- A force in the same direction as the motion does positive work, i.e., transfers energy into the object on which it acts.
- A force in the opposite direction compared to the motion does negative work, i.e., transfers energy out of the object on which it acts.
- When there is no motion, no mechanical work is done. The human body burns calories when it exerts a force without moving, but this is an internal energy transfer of energy within the body, and thus does not fall within the scientific definition of work.
- A force perpendicular to the motion does no work.

When the force is not constant, the above equation should be generalized as an integral, $\int F_\parallel \, dx$.

There is only one meaningful (rotationally invariant) way of defining a multiplication of vectors whose result is a scalar, and it is known as the vector dot product:

$$\mathbf{b} \cdot \mathbf{c} = b_x c_x + b_y c_y + b_z c_z$$
$$= |\mathbf{b}| |\mathbf{c}| \cos \theta_{bc}.$$

The dot product has most of the usual properties associated with multiplication, except that there is no “dot division.” The dot product can be used to compute mechanical work as $W = \mathbf{F} \cdot \mathbf{d}$.

Machines such as pulleys, levers, and gears may increase or decrease a force, but they can never increase or decrease the amount of work done. That would violate conservation of energy unless the
machine had some source of stored energy or some way to accept and store up energy.

There are some situations in which the equation \( W = F \cdot |d| \) is ambiguous or not true, and these issues are discussed rigorously in section 13.6. However, problems can usually be avoided by analyzing the types of energy being transferred before plunging into the math. In any case there is no substitute for a physical understanding of the processes involved.

The techniques developed for calculating work can also be applied to the calculation of potential energy. We fix some position as a reference position, and calculate the potential energy for some other position, \( x \), as

\[
PE_x = -W_{\text{ref} \rightarrow x}.
\]

The following two equations for potential energy have broader significance than might be suspected based on the limited situations in which they were derived:

\[
PE = \frac{1}{2} k (x - x_0)^2.
\]

[potential energy of a spring having spring constant \( k \), when stretched or compressed from the equilibrium position \( x_0 \); analogous equations apply for the twisting, bending, compression, or stretching of any object.]

\[
PE = -\frac{GMm}{r}
\]

[gravitational potential energy of objects of masses \( M \) and \( m \), separated by a distance \( r \); an analogous equation applies to the electrical potential energy of an electron in an atom.]
Problems

Key
✓ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 Two speedboats are identical, but one has more people aboard than the other. Although the total masses of the two boats are unequal, suppose that they happen to have the same kinetic energy. In a boat, as in a car, it’s important to be able to stop in time to avoid hitting things. (a) If the frictional force from the water is the same in both cases, how will the boats’ stopping distances compare? Explain. (b) Compare the times required for the boats to stop.

2 In each of the following situations, is the work being done positive, negative, or zero? (a) a bull paws the ground; (b) a fishing boat pulls a net through the water behind it; (c) the water resists the motion of the net through it; (d) you stand behind a pickup truck and lower a bale of hay from the truck’s bed to the ground. Explain. [Based on a problem by Serway and Faughn.]

3 (a) Suppose work is done in one-dimensional motion. What happens to the work if you reverse the direction of the positive coordinate axis? Base your answer directly on the definition of work as a transfer of mechanical energy. (b) Now answer the question based on the \( W = Fd \) rule.

4 Does it make sense to say that work is conserved?

5 A microwave oven works by twisting molecules one way and then the other, counterclockwise and then clockwise about their own centers, millions of times a second. If you put an ice cube or a stick of butter in a microwave, you’ll observe that the solid doesn’t heat very quickly, although eventually melting begins in one small spot. Once this spot forms, it grows rapidly, while the rest of the solid remains solid; it appears that a microwave oven heats a liquid much more rapidly than a solid. Explain why this should happen, based on the atomic-level description of heat, solids, and liquids. (See, e.g., figure b on page 341.)

Don’t repeat the following common mistakes:

*In a solid, the atoms are packed more tightly and have less space between them.* Not true. Ice floats because it’s less dense than water.

*In a liquid, the atoms are moving much faster.* No, the difference in average speed between ice at \(-1\)°C and water at 1°C is only 0.4%. 

A bull paws the ground, as in problem 2.
Most modern bow hunters in the U.S. use a fancy mechanical bow called a compound bow, which looks nothing like what most people imagine when they think of a bow and arrow. It has a system of pulleys designed to produce the force curve shown in the figure, where \( F \) is the force required to pull the string back, and \( x \) is the distance between the string and the center of the bow’s body. It is not a linear Hooke’s-law graph, as it would be for an old-fashioned bow. The big advantage of the design is that relatively little force is required to hold the bow stretched to point B on the graph. This is the force required from the hunter in order to hold the bow ready while waiting for a shot. Since it may be necessary to wait a long time, this force can’t be too big. An old-fashioned bow, designed to require the same amount of force when fully drawn, would shoot arrows at much lower speeds, since its graph would be a straight line from A to B. For the graph shown in the figure (taken from realistic data), find the speed at which a 26 g arrow is released, assuming that 70% of the mechanical work done by the hand is actually transmitted to the arrow. (The other 30% is lost to frictional heating inside the bow and kinetic energy of the recoiling and vibrating bow.)

In the power stroke of a car’s gasoline engine, the fuel-air mixture is ignited by the spark plug, explodes, and pushes the piston out. The exploding mixture’s force on the piston head is greatest at the beginning of the explosion, and decreases as the mixture expands. It can be approximated by \( F = \frac{a}{x} \), where \( x \) is the distance from the cylinder to the piston head, and \( a \) is a constant with units of N·m. (Actually \( a/x^{1.4} \) would be more accurate, but the problem works out more nicely with \( a/x \)!) The piston begins its stroke at \( x = x_1 \), and ends at \( x = x_2 \).

(a) Find the amount of work done in one stroke by one cylinder.

(b) The 1965 Rambler had six cylinders, each with \( a = 220 \text{ N·m} \), \( x_1 = 1.2 \text{ cm} \), and \( x_2 = 10.2 \text{ cm} \). Assume the engine is running at 4800 r.p.m., so that during one minute, each of the six cylinders performs 2400 power strokes. (Power strokes only happen every other revolution.) Find the engine’s power, in units of horsepower (1 hp=746 W).

(c) The compression ratio of an engine is defined as \( x_2/x_1 \). Explain in words why the car’s power would be exactly the same if \( x_1 \) and \( x_2 \) were, say, halved or tripled, maintaining the same compression ratio of 8.5. Explain why this would not quite be true with the more realistic force equation \( F = a/x^{1.4} \).
The figure, redrawn from Gray’s Anatomy, shows the tension of which a muscle is capable. The variable $x$ is defined as the contraction of the muscle from its maximum length $L$, so that at $x = 0$ the muscle has length $L$, and at $x = L$ the muscle would theoretically have zero length. In reality, the muscle can only contract to $x = cL$, where $c$ is less than 1. When the muscle is extended to its maximum length, at $x = 0$, it is capable of the greatest tension, $T_0$. As the muscle contracts, however, it becomes weaker. Gray suggests approximating this function as a linear decrease, which would theoretically extrapolate to zero at $x = L$. 

(a) Find the maximum work the muscle can do in one contraction, in terms of $c$, $L$, and $T_0$.

(b) Show that your answer to part a has the right units.

(c) Show that your answer to part a has the right behavior when $c = 0$ and when $c = 1$.

(d) Gray also states that the absolute maximum tension $T_0$ has been found to be approximately proportional to the muscle’s cross-sectional area $A$ (which is presumably measured at $x = 0$), with proportionality constant $k$. Approximating the muscle as a cylinder, show that your answer from part a can be reexpressed in terms of the volume, $V$, eliminating $L$ and $A$.

(e) Evaluate your result numerically for a biceps muscle with a volume of 200 cm$^3$, with $c = 0.8$ and $k = 100$ N/cm$^2$ as estimated by Gray.

In the earth’s atmosphere, the molecules are constantly moving around. Because temperature is a measure of kinetic energy per molecule, the average kinetic energy of each type of molecule is the same, e.g., the average KE of the O$_2$ molecules is the same as the average KE of the N$_2$ molecules. (a) If the mass of an O$_2$ molecule is eight times greater than that of a He atom, what is the ratio of their average speeds? Which way is the ratio, i.e., which is typically moving faster? (b) Use your result from part a to explain why any helium occurring naturally in the atmosphere has long since escaped into outer space, never to return. (Helium is obtained commercially by extracting it from rocks.) You may want to do problem 12 first, for insight.

Weiping lifts a rock with a weight of 1.0 N through a height of 1.0 m, and then lowers it back down to the starting point. Bubba pushes a table 1.0 m across the floor at constant speed, requiring a force of 1.0 N, and then pushes it back to where it started. (a) Compare the total work done by Weiping and Bubba. (b) Check that your answers to part a make sense, using the definition of work: work is the transfer of energy. In your answer, you’ll need to discuss what specific type of energy is involved in each case.
11 In one of his more flamboyant moments, Galileo wrote “Who does not know that a horse falling from a height of three or four cubits will break his bones, while a dog falling from the same height or a cat from a height of eight or ten cubits will suffer no injury? Equally harmless would be the fall of a grasshopper from a tower or the fall of an ant from the distance of the moon.” Find the speed of an ant that falls to earth from the distance of the moon at the moment when it is about to enter the atmosphere. Assume it is released from a point that is not actually near the moon, so the moon’s gravity is negligible. You will need the result of example 10 on p. 374.

\[ \sqrt{12} \]

Starting at a distance \( r \) from a planet of mass \( M \), how fast must an object be moving in order to have a hyperbolic orbit, i.e., one that never comes back to the planet? This velocity is called the escape velocity. Interpreting the result, does it matter in what direction the velocity is? Does it matter what mass the object has? Does the object escape because it is moving too fast for gravity to act on it?

\[ \sqrt{13} \]

A projectile is moving directly away from a planet of mass \( M \) at exactly escape velocity. (a) Find \( r \), the distance from the projectile to the center of the planet, as a function of time, \( t \), and also find \( v(t) \). \( \checkmark \)
(b) Check the units of your answer.
(c) Does \( v \) show the correct behavior as \( t \) approaches infinity?
\( \triangleright \) Hint, p. 548

\[ \sqrt{14} \]

A car starts from rest at \( t = 0 \), and starts speeding up with constant acceleration. (a) Find the car’s kinetic energy in terms of its mass, \( m \), acceleration, \( a \), and the time, \( t \). (b) Your answer in the previous part also equals the amount of work, \( W \), done from \( t = 0 \) until time \( t \). Take the derivative of the previous expression to find the power expended by the car at time \( t \). (c) Suppose two cars with the same mass both start from rest at the same time, but one has twice as much acceleration as the other. At any moment, how many times more power is being dissipated by the more quickly accelerating car? (The answer is not 2.) \( \checkmark \)
A car accelerates from rest. At low speeds, its acceleration is limited by static friction, so that if we press too hard on the gas, we will “burn rubber” (or, for many newer cars, a computerized traction-control system will override the gas pedal). At higher speeds, the limit on acceleration comes from the power of the engine, which puts a limit on how fast kinetic energy can be developed.

(a) Show that if a force $F$ is applied to an object moving at speed $v$, the power required is given by $P = vF$.

(b) Find the speed $v$ at which we cross over from the first regime described above to the second. At speeds higher than this, the engine does not have enough power to burn rubber. Express your result in terms of the car’s power $P$, its mass $m$, the coefficient of static friction $\mu_s$, and $g$.

(c) Show that your answer to part b has units that make sense.

(d) Show that the dependence of your answer on each of the four variables makes sense physically.

(e) The 2010 Maserati Gran Turismo Convertible has a maximum power of $3.23 \times 10^5$ W (433 horsepower) and a mass (including a 50-kg driver) of $2.03 \times 10^3$ kg. (This power is the maximum the engine can supply at its optimum frequency of 7600 r.p.m. Presumably the automatic transmission is designed so a gear is available in which the engine will be running at very nearly this frequency when the car is moving at $v$.) Rubber on asphalt has $\mu_s \approx 0.9$. Find $v$ for this car. Answer: 18 m/s, or about 40 miles per hour.

(f) Our analysis has neglected air friction, which can probably be approximated as a force proportional to $v^2$. The existence of this force is the reason that the car has a maximum speed, which is 176 miles per hour. To get a feeling for how good an approximation it is to ignore air friction, find what fraction of the engine’s maximum power is being used to overcome air resistance when the car is moving at the speed $v$ found in part e. Answer: 1%

In 1935, Yukawa proposed an early theory of the force that held the neutrons and protons together in the nucleus. His equation for the potential energy of two such particles, at a center-to-center distance $r$, was $PE(r) = gr^{-1}e^{-r/a}$, where $g$ parametrizes the strength of the interaction, $e$ is the base of natural logarithms, and $a$ is about $10^{-15}$ m. Find the force between two nucleons that would be consistent with this equation for the potential energy.

The magnitude of the force between two magnets separated by a distance $r$ can be approximated as $kr^{-3}$ for large values of $r$. The constant $k$ depends on the strengths of the magnets and the relative orientations of their north and south poles. Two magnets are released on a slippery surface at an initial distance $r_i$, and begin sliding towards each other. What will be the total kinetic energy of the two magnets when they reach a final distance $r_f$? (Ignore friction.)
A rail gun is a device like a train on a track, with the train propelled by a powerful electrical pulse. Very high speeds have been demonstrated in test models, and rail guns have been proposed as an alternative to rockets for sending into outer space any object that would be strong enough to survive the extreme accelerations. Suppose that the rail gun capsule is launched straight up, and that the force of air friction acting on it is given by $F = be^{-cx}$, where $x$ is the altitude, $b$ and $c$ are constants, and $e$ is the base of natural logarithms. The exponential decay occurs because the atmosphere gets thinner with increasing altitude. (In reality, the force would probably drop off even faster than an exponential, because the capsule would be slowing down somewhat.) Find the amount of kinetic energy lost by the capsule due to air friction between when it is launched and when it is completely beyond the atmosphere. (Gravity is negligible, since the air friction force is much greater than the gravitational force.)

A certain binary star system consists of two stars with masses $m_1$ and $m_2$, separated by a distance $b$. A comet, originally nearly at rest in deep space, drops into the system and at a certain point in time arrives at the midpoint between the two stars. For that moment in time, find its velocity, $v$, symbolically in terms of $b$, $m_1$, $m_2$, and fundamental constants.

Find the angle between the following two vectors:

\[
\hat{x} + 2\hat{y} + 3\hat{z} \\
4\hat{x} + 5\hat{y} + 6\hat{z}
\]

▷ Hint, p. 548

An airplane flies in the positive direction along the $x$ axis, through crosswinds that exert a force $\mathbf{F} = (a + bx)\hat{x} + (c + dx)\hat{y}$. Find the work done by the wind on the plane, and by the plane on the wind, in traveling from the origin to position $x$.

Prove that the dot product defined in section 13.3 is rotationally invariant in the sense of section 7.5.

Fill in the details of the proof of $\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$ on page 371.
24 A space probe of mass \( m \) is dropped into a previously unexplored spherical cloud of gas and dust, and accelerates toward the center of the cloud under the influence of the cloud's gravity. Measurements of its velocity allow its potential energy, \( P_E \), to be determined as a function of the distance \( r \) from the cloud's center. The mass in the cloud is distributed in a spherically symmetric way, so its density, \( \rho(r) \), depends only on \( r \) and not on the angular coordinates. Show that by finding \( P_E \), one can infer \( \rho(r) \) as follows:

\[
\rho(r) = \frac{1}{4\pi Gmr^2} \frac{d}{dr} \left( r^2 \frac{dP_E}{dr} \right).
\]

25 The purpose of this problem is to estimate the height of the tides. The main reason for the tides is the moon’s gravity, and we’ll neglect the effect of the sun. Also, real tides are heavily influenced by landforms that channel the flow of water, but we’ll think of the earth as if it was completely covered with oceans. Under these assumptions, the ocean surface should be a surface of constant \( U/m \). That is, a thimbleful of water, \( m \), should not be able to gain or lose any gravitational energy by moving from one point on the ocean surface to another. If only the spherical earth’s gravity was present, then we’d have \( U/m = -GM_e/r \), and a surface of constant \( U/m \) would be a surface of constant \( r \), i.e., the ocean’s surface would be spherical. Taking into account the moon’s gravity, the main effect is to shift the center of the sphere, but the sphere also becomes slightly distorted into an approximately ellipsoidal shape. (The shift of the center is not physically related to the tides, since the solid part of the earth tends to be centered within the oceans; really, this effect has to do with the motion of the whole earth through space, and the way that it wobbles due to the moon’s gravity.) Determine the amount by which the long axis of the ellipsoid exceeds the short axis.

\[
> \text{Hint, p. 548}
\]

26 A mass moving in one dimension is attached to a horizontal spring. It slides on the surface below it, with equal coefficients of static and kinetic friction, \( \mu_k = \mu_s \). The equilibrium position is \( x = 0 \). If the mass is pulled to some initial position and released from rest, it will complete some number of oscillations before friction brings it to a stop. When released from \( x = a \) (\( a > 0 \)), it completes exactly 1/4 of an oscillation, i.e., it stops precisely at \( x = 0 \). Similarly, define \( b > 0 \) as the greatest \( x \) from which it could be released and complete 1/2 of an oscillation, stopping on the far side and not coming back toward equilibrium. Find \( b/a \). Hint: To keep the algebra simple, set every fixed parameter of the system equal to 1.
“Big wall” climbing is a specialized type of rock climbing that involves going up tall cliffs such as the ones in Yosemite, usually with the climbers spending at least one night sleeping on a natural ledge or an artificial “portaledge.” In this style of climbing, each pitch of the climb involves strenuously hauling up several heavy bags of gear — a fact that has caused these climbs to be referred to as “vertical ditch digging.” (a) If an 80 kg haul bag has to be pulled up the full length of a 60 m rope, how much work is done? (b) Since it can be difficult to lift 80 kg, a 2:1 pulley is often used. The hauler then lifts the equivalent of 40 kg, but has to pull in 120 m of rope. How much work is done in this case?

Let \( a \) and \( b \) be any two numbers (not both zero), and let \( \mathbf{u} = a\mathbf{x} + b\mathbf{y} \). Suppose we want to find a (nonzero) second vector \( \mathbf{v} \) in the \( x-y \) plane that is perpendicular to \( \mathbf{u} \). Use the vector dot product to write down a condition for \( \mathbf{v} \) to satisfy, find a suitable \( \mathbf{v} \), and check using the dot product that it is indeed a solution.

A soccer ball of mass \( m \) is moving at speed \( v \) when you kick it in the same direction it is moving. You kick it with constant force \( F \), and you want to triple the ball’s speed. Over what distance must your foot be in contact with the ball? [problem by B. Shotwell]
In many subfields of physics these days, it is possible to read an entire issue of a journal without ever encountering an equation involving force or a reference to Newton’s laws of motion. In the last hundred and fifty years, an entirely different framework has been developed for physics, based on conservation laws.

The new approach is not just preferred because it is in fashion. It applies inside an atom or near a black hole, where Newton’s laws do not. Even in everyday situations the new approach can be superior. We have already seen how perpetual motion machines could be designed that were too complex to be easily debunked by Newton’s laws. The beauty of conservation laws is that they tell us something must remain the same, regardless of the complexity of the process.

So far we have discussed only two conservation laws, the laws of conservation of mass and energy. Is there any reason to believe that further conservation laws are needed in order to replace Newton’s laws as a complete description of nature? Yes. Conservation of mass and energy do not relate in any way to the three dimensions of space, because both are scalars. Conservation of energy, for instance, does not prevent the planet earth from abruptly making a 90-degree turn and heading straight into the sun, because kinetic energy does not depend on direction. In this chapter, we develop a new conserved quantity, called momentum, which is a vector.
14.1 Momentum

A conserved quantity of motion

Your first encounter with conservation of momentum may have come as a small child unjustly confined to a shopping cart. You spot something interesting to play with, like the display case of imported wine down at the end of the aisle, and decide to push the cart over there. But being imprisoned by Dad in the cart was not the only injustice that day. There was a far greater conspiracy to thwart your young id, one that originated in the laws of nature. Pushing forward did nudge the cart forward, but it pushed you backward. If the wheels of the cart were well lubricated, it wouldn’t matter how you jerked, yanked, or kicked off from the back of the cart. You could not cause any overall forward motion of the entire system consisting of the cart with you inside.

In the Newtonian framework, we describe this as arising from Newton’s third law. The cart made a force on you that was equal and opposite to your force on it. In the framework of conservation laws, we cannot attribute your frustration to conservation of energy. It would have been perfectly possible for you to transform some of the internal chemical energy stored in your body to kinetic energy of the cart and your body.

The following characteristics of the situation suggest that there may be a new conservation law involved:

A closed system is involved. All conservation laws deal with closed systems. You and the cart are a closed system, since the well-oiled wheels prevent the floor from making any forward force on you.

Something remains unchanged. The overall velocity of the system started out being zero, and you cannot change it. This vague reference to “overall velocity” can be made more precise: it is the velocity of the system’s center of mass that cannot be changed.

Something can be transferred back and forth without changing the total amount. If we define forward as positive and backward as negative, then one part of the system can gain positive motion if another part acquires negative motion. If we don’t want to worry about positive and negative signs, we can imagine that the whole cart was initially gliding forward on its well-oiled wheels. By kicking off from the back of the cart, you could increase your own velocity, but this inevitably causes the cart to slow down.
It thus appears that there is some numerical measure of an object’s quantity of motion that is conserved when you add up all the objects within a system.

**Momentum**

Although velocity has been referred to, it is not the total velocity of a closed system that remains constant. If it was, then firing a gun would cause the gun to recoil at the same velocity as the bullet! The gun does recoil, but at a much lower velocity than the bullet. Newton’s third law tells us

\[ F_{\text{gun on bullet}} = -F_{\text{bullet on gun}}, \]

and assuming a constant force for simplicity, Newton’s second law allows us to change this to

\[ m_{\text{bullet}} \frac{\Delta v_{\text{bullet}}}{\Delta t} = -m_{\text{gun}} \frac{\Delta v_{\text{gun}}}{\Delta t}. \]

Thus if the gun has 100 times more mass than the bullet, it will recoil at a velocity that is 100 times smaller and in the opposite direction, represented by the opposite sign. The quantity \( mv \) is therefore apparently a useful measure of motion, and we give it a name, momentum, and a symbol, \( p \). (As far as I know, the letter “p” was just chosen at random, since “m” was already being used for mass.) The situations discussed so far have been one-dimensional, but in three-dimensional situations it is treated as a vector.

**Definition of momentum for material objects**

The momentum of a material object, i.e., a piece of matter, is defined as

\[ p = mv, \]

the product of the object’s mass and its velocity vector.

The units of momentum are kg·m/s, and there is unfortunately no abbreviation for this clumsy combination of units.

The reasoning leading up to the definition of momentum was all based on the search for a conservation law, and the only reason why we bother to define such a quantity is that experiments show it is conserved:

**The law of conservation of momentum**

In any closed system, the vector sum of all the momenta remains constant,

\[ p_{1i} + p_{2i} + \ldots = p_{1f} + p_{2f} + \ldots, \]

where \( i \) labels the initial and \( f \) the final momenta. (A closed system is one on which no external forces act.)
This chapter first addresses the one-dimensional case, in which the direction of the momentum can be taken into account by using plus and minus signs. We then pass to three dimensions, necessitating the use of vector addition.

A subtle point about conservation laws is that they all refer to “closed systems,” but “closed” means different things in different cases. When discussing conservation of mass, “closed” means a system that doesn’t have matter moving in or out of it. With energy, we mean that there is no work or heat transfer occurring across the boundary of the system. For momentum conservation, “closed” means there are no external forces reaching into the system.

A cannon example 1
A cannon of mass 1000 kg fires a 10-kg shell at a velocity of 200 m/s. At what speed does the cannon recoil?

The law of conservation of momentum tells us that

\[ p_{\text{cannon},i} + p_{\text{shell},i} = p_{\text{cannon},f} + p_{\text{shell},f}. \]

Choosing a coordinate system in which the cannon points in the positive direction, the given information is

\[ p_{\text{cannon},i} = 0 \]
\[ p_{\text{shell},i} = 0 \]
\[ p_{\text{shell},f} = 2000 \text{ kg} \cdot \text{m/s}. \]

We must have \( p_{\text{cannon},f} = -2000 \text{ kg} \cdot \text{m/s}, \) so the recoil velocity of the cannon is \(-2 \text{ m/s}.\)

Ion drive for propelling spacecraft example 2
The experimental solar-powered ion drive of the Deep Space 1 space probe expels its xenon gas exhaust at a speed of 30,000 m/s, ten times faster than the exhaust velocity for a typical chemical-fuel rocket engine. Roughly how many times greater is the maximum speed this spacecraft can reach, compared with a chemical-fueled probe with the same mass of fuel (“reaction mass”) available for pushing out the back as exhaust?

Momentum equals mass multiplied by velocity. Both spacecraft are assumed to have the same amount of reaction mass, and the ion drive’s exhaust has a velocity ten times greater, so the momentum of its exhaust is ten times greater. Before the engine starts firing, neither the probe nor the exhaust has any momentum, so the total momentum of the system is zero. By conservation of momentum, the total momentum must also be zero after
The ion drive engine of the NASA Deep Space 1 probe, shown under construction (left) and being tested in a vacuum chamber (right) prior to its October 1998 launch. Intended mainly as a test vehicle for new technologies, the craft nevertheless carried out a successful scientific program that included a flyby of a comet.

all the exhaust has been expelled. If we define the positive direction as the direction the spacecraft is going, then the negative momentum of the exhaust is canceled by the positive momentum of the spacecraft. The ion drive allows a final speed that is ten times greater. (This simplified analysis ignores the fact that the reaction mass expelled later in the burn is not moving backward as fast, because of the forward speed of the already-moving spacecraft.)

**Generalization of the momentum concept**

As with all the conservation laws, the law of conservation of momentum has evolved over time. In the 1800’s it was found that a beam of light striking an object would give it some momentum, even though light has no mass, and would therefore have no momentum.
Steam and other gases boiling off of the nucleus of Halley’s comet. This close-up photo was taken by the European Giotto space probe, which passed within 596 km of the nucleus on March 13, 1986.

Halley’s comet, in a much less magnified view from a ground-based telescope.

Momentum is not always equal to \(mv\). Like many comets, Halley’s comet has a very elongated elliptical orbit. About once per century, its orbit brings it close to the sun. The comet’s head, or nucleus, is composed of dirty ice, so the energy deposited by the intense sunlight boils off steam and dust, b. The sunlight does not just carry energy, however — it also carries momentum. The momentum of the sunlight impacting on the smaller dust particles pushes them away from the sun, forming a tail, c. By analogy with matter, for which momentum equals \(mv\), you would expect that massless light would have zero momentum, but the equation \(p = mv\) is not the correct one for light, and light does have momentum. (The gases typically form a second, distinct tail whose motion is controlled by the sun’s magnetic field.)

The reason for bringing this up is not so that you can plug numbers into formulas in these exotic situations. The point is that the conservation laws have proven so sturdy exactly because they can easily be amended to fit new circumstances. Newton’s laws are no longer at the center of the stage of physics because they did not have the same adaptability. More generally, the moral of this story is the provisional nature of scientific truth.

It should also be noted that conservation of momentum is not a consequence of Newton’s laws, as is often asserted in textbooks. Newton’s laws do not apply to light, and therefore could not possibly be used to prove anything about a concept as general as the conservation of momentum in its modern form.

Momentum compared to kinetic energy

Momentum and kinetic energy are both measures of the quantity of motion, and a sideshow in the Newton-Leibnitz controversy over who invented calculus was an argument over whether \(mv\) (i.e., momentum) or \(mv^2\) (i.e., kinetic energy without the 1/2 in front)
was the “true” measure of motion. The modern student can cer-
tainly be excused for wondering why we need both quantities, when
their complementary nature was not evident to the greatest minds
of the 1700’s. The following table highlights their differences.

<table>
<thead>
<tr>
<th>kinetic energy . . .</th>
<th>momentum . . .</th>
</tr>
</thead>
<tbody>
<tr>
<td>is a scalar.</td>
<td>is a vector.</td>
</tr>
<tr>
<td>is not changed by a force perpendicular to the motion, which changes only the direction of the velocity vector.</td>
<td>is changed by any force, since a change in either the magnitude or the direction of the velocity vector will result in a change in the momentum vector.</td>
</tr>
<tr>
<td>is always positive, and cannot cancel out.</td>
<td>cancels with momentum in the opposite direction.</td>
</tr>
<tr>
<td>can be traded for other forms of energy that do not involve motion. KE is not a conserved quantity by itself.</td>
<td>is always conserved in a closed system.</td>
</tr>
<tr>
<td>is quadrupled if the velocity is doubled.</td>
<td>is doubled if the velocity is doubled.</td>
</tr>
</tbody>
</table>

A spinning top example 4
A spinning top has zero total momentum, because for every mov-
ing point, there is another point on the opposite side that cancels its momentum. It does, however, have kinetic energy.

Why a tuning fork has two prongs example 5
A tuning fork is made with two prongs so that they can vibrate in opposite directions, canceling their momenta. In a hypothetical version with only one prong, the momentum would have to oscillate, and this momentum would have to come from somewhere, such as the hand holding the fork. The result would be that vibrations would be transmitted to the hand and rapidly die out. In a two-prong fork, the two momenta cancel, but the energies don’t.

Momentum and kinetic energy in firing a rifle example 6
The rifle and bullet have zero momentum and zero kinetic energy to start with. When the trigger is pulled, the bullet gains some momentum in the forward direction, but this is canceled by the rifle’s backward momentum, so the total momentum is still zero. The kinetic energies of the gun and bullet are both positive scalars, however, and do not cancel. The total kinetic energy is allowed to increase, because kinetic energy is being traded for other forms of energy. Initially there is chemical energy in the gunpowder. This chemical energy is converted into heat, sound, and kinetic energy. The gun’s “backward” kinetic energy does not refrigerate the shooter’s shoulder!
The wobbly earth example 7
As the moon completes half a circle around the earth, its motion reverses direction. This does not involve any change in kinetic energy, and the earth’s gravitational force does not do any work on the moon. The reversed velocity vector does, however, imply a reversed momentum vector, so conservation of momentum in the closed earth-moon system tells us that the earth must also change its momentum. In fact, the earth wobbles in a little “orbit” about a point below its surface on the line connecting it and the moon. The two bodies’ momentum vectors always point in opposite directions and cancel each other out.

The earth and moon get a divorce example 8
Why can’t the moon suddenly decide to fly off one way and the earth the other way? It is not forbidden by conservation of momentum, because the moon’s newly acquired momentum in one direction could be canceled out by the change in the momentum of the earth, supposing the earth headed the opposite direction at the appropriate, slower speed. The catastrophe is forbidden by conservation of energy, because both their energies would have to increase greatly.

Momentum and kinetic energy of a glacier example 9
A cubic-kilometer glacier would have a mass of about $10^{12}$ kg. If it moves at a speed of $10^{-5}$ m/s, then its momentum is $10^{7}$ kg·m/s. This is the kind of heroic-scale result we expect, perhaps the equivalent of the space shuttle taking off, or all the cars in LA driving in the same direction at freeway speed. Its kinetic energy, however, is only 50 J, the equivalent of the calories contained in a poppy seed or the energy in a drop of gasoline too small to be seen without a microscope. The surprisingly small kinetic energy is because kinetic energy is proportional to the square of the velocity, and the square of a small number is an even smaller number.

Discussion questions
A If all the air molecules in the room settled down in a thin film on the floor, would that violate conservation of momentum? Conservation of energy?
B A refrigerator has coils in the back that get hot, and heat is molecular motion. These moving molecules have both energy and momentum. Why doesn’t the refrigerator need to be tied to the wall to keep it from recoiling from the momentum it loses out the back?