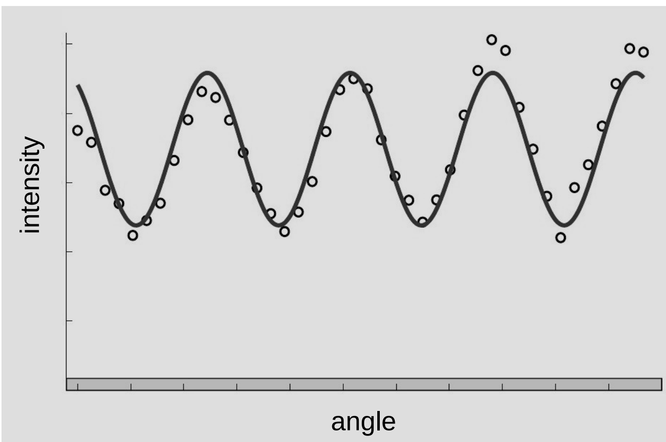
Practice Exam 5 for Physics 223

Useful Data

gravitational constant	$G = 6.67 \times 10^{-11}$ N·m^2/kg^2
Coulomb constant	$k = 8.99 \times 10^9$ J·m/C^2 or N·m^2/C^2
Boltzmann's constant	$k = 1.38 \times 10^{-23}$ J/K
quantum of charge	$e = 1.60 \times 10^{-19}$ C
speed of light	$c = 3.00 \times 10^8$ m/s
Planck's constant	$h = 6.63 \times 10^{-34}$ J·s
absolute zero	$-273.15\,°$C
speed of sound in air	340 m/s
mass of electron	9.11×10^{-31} kg

1 Various physicists have attempted to diffract larger and larger objects, and it has even been proposed that it might be possible to diffract a virus in the near future. So far, the largest object ever diffracted, by Eibenberger *et al.* in 2013, was a large fluorocarbon molecule containing 810 atoms. The diffraction pattern is shown in the figure (CC-BY license). The circles are the experimental data points. The curve is a fit to the data. The dark strip at the bottom shows the level of background signals.

The molecule used in this experiment had a mass of 1.6810×10^{-23} kg. A beam of these molecules was prepared with a velocity of 85 m/s, and the beam was sent through a diffraction grating with a spacing of 266 nm. Find the angular spacing of the resulting diffraction pattern, in radians.



2 We have found that the Schrödinger equation for a particle in a box in one dimension has solutions that are sine waves. Show that there is an additional possibility, which we never discussed: a linear solution $\Psi(x) = ax + b$, where a and b are constants. Why would this not be observed physically?

3 The following function is a one-dimensional model of the wavefunction of an electron in a diatomic molecule that extends from $x = -L$ to $x = L$.

$$\Psi(x) = \begin{cases} 2A\sin(\pi x/L) & \text{if } -L \le x \le 0 \\ A\sin(2\pi x/L) & \text{if } 0 \le x \le L \\ 0 & \text{elsewhere} \end{cases}$$

The wavelength is different in the two halves of the molecule because the electron feels a more attractive potential in the right-hand side, so it has more kinetic energy. The multiplicative factor of 2 for $-L \le x \le 0$ is chosen so that the wavefunction is differentiable at $x = 0$. Determine the constant A.

4 Find the two mistakes in the following attempt to calculate the ground-state energy of a particle in a box in one dimension.

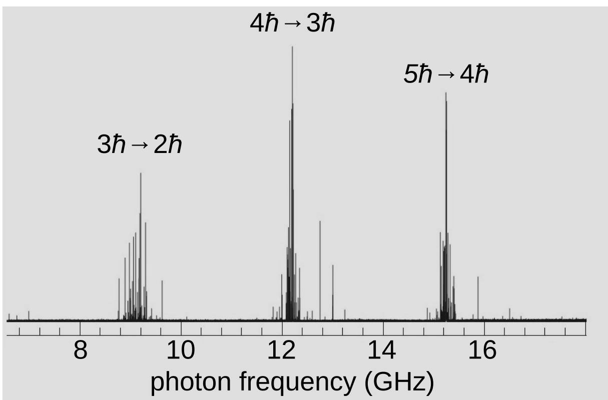
wrong: *A particle of mass m is in a box of length L. The ground state has wavelength $\lambda = L$. The frequency is related to the wavelength by $c = f\lambda$, so $f = c/\lambda = c/L$. The energy is $E = hf = hc/L$.*

——————> *Turn the page.*

5 Classically, the kinetic energy of a rotating rigid body is $E = L^2/2I$, where L is the angular momentum and I the moment of inertia. Surprisingly, only a slight alteration is needed in order to find the corresponding formula that is valid in quantum mechanics:

$$E = \frac{L(L+\hbar)}{2I}.$$

The angular momentum L is also quantized in units of $\hbar$, as always for orbital angular momentum in quantum mechanics.
(a) Show that this modification to the equation is consistent with the correspondence principle.
(b) The figure shows the spectrum of photons emitted by the molecule CF_3I (Wikipedia user Nnrw, CC-BY-SA license). Each of the three peaks in the histogram is split into a somewhat messy group of sub-peaks, due to interactions with the atomic nuclei. Ignoring this complication, each peak can be interpreted as in the labels in the diagram, as the photon emitted by the molecule in making a transition from the state with angular momentum L to the one with $L - \hbar$. Notice that the three peaks appear to be evenly spaced, like a picket fence. Explain why this happens based on the energy formula given above.



6 Compute the following.
(a) $|i(1+i)(-1-i)(-2i)|$
(b) $\arg[i(1+i)(-1-i)(-2i)]$
(c) $\ln(-1)$

Answer to problem 1

The wavelength is $\lambda = h/p = h/mv$. The angular spacing of the diffraction pattern, in radians, is $\Delta\theta = \lambda/d = h/mvd = 1.7 \times 10^{-6}$. This is small, but that makes sense. The angle is inversely proportional to m, so if your goal is to measure a diffraction pattern for a large object, you're going to end up having to measure very small angles.

Answer to problem 2

The Schrödinger equation is

$$E\Psi = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + U\Psi.$$

For the given wavefunction we have $\Psi' = a$ and $\Psi'' = 0$, so the Schrödinger equation becomes $E\Psi = U\Psi$. This can be satisfied if $E = U$, so the given wavefunction is a solution with energy equal to U.

Physically, we would not observe this solution because for a particle in a box, we need wavefunctions that go to zero at the walls of the box. The constants a and b could be adjusted to make $\Psi = 0$ at one wall, but not at both walls, unless $a = 0$ and $b = 0$. The solution with $a = 0$ and $b = 0$ is simply the trivial solution to the Schrödinger equation $\Psi = 0$, which is never physically meaningful because we can't normalize it.

Answer to problem 3

Normalization requires the following.

$$\begin{aligned} 1 &= \int_{-L}^{L} \Psi^2 dx \\ &= 4A^2 \int_{-L}^{0} \sin^2(\pi x/L) dx + A^2 \int_{0}^{L} \sin^2(2\pi x/L) dx \\ A^{-2} &= 4 \int_{-L}^{0} \sin^2(\pi x/L) dx + \int_{0}^{L} \sin^2(2\pi x/L) dx \end{aligned}$$

In each integral, the integrand has an average value of 1/2, so

$$\begin{aligned} A^{-2} &= 4(L)(1/2) + (L)(1/2) \\ &= 5L/2 \\ A &= \sqrt{2/5L}. \end{aligned}$$

Answer to problem 4

Error 1: The longest wavelength comes when we make a standing wave pattern with one hump. In this wave pattern, *half* a wavelength fits in the box, not a whole wavelength.

Error 2: It's not true that $c = f\lambda$. This particle has mass m, which is presumably not zero, so it doesn't go at the speed of light. The right method would have been to calculate $p = h/\lambda$, then find the kinetic energy using $K = (1/2)mv^2$ (assuming the motion is nonrelativistic).

Answer to problem 5

(a) The correspondence principle requires that the new equation agree with the old one to a good approximation, in the kind of classical experiments in which the old one had already been verified. In a classical experiment, the angular momentum L is a macroscopic number, so it is much greater than $\hbar$. Therefore replacing $L + \hbar$ with L is essentially a perfect approximation in the classical limit.

(b) The key concept is that the energy of the photon equals the difference in energy between two energy levels of the molecule. For convenience, let's work in units such that $\hbar = 1$; if the spectrum is equally spaced, then this is so regardless of what system of units we use. We then have

$$\begin{aligned} E_{\text{photon}} &= E_L - E_{L-1} \\ &= (1/2I)[L(L+1) - (L-1)L] \\ &= \frac{L}{I}. \end{aligned}$$

The difference between the energy of one photon and the next one in the series is $1/I$, which is always the same. If the energies are equally spaced, then so are the frequencies $f = E/h$.

Answer to problem 6
(a) The magnitude of the product is the product of the magnitudes of the factors, $(1)(\sqrt{2})(\sqrt{2})(2) = 4$.
(b) The argument of the product is the sum of the arguments, $\pi/2 + \pi/4 + 5\pi/4 + 3\pi/2 = 7\pi/2$, which is equivalent to $-\pi/2$.
(c) We want a complex number z such that $e^z = -1$. Euler's formula gives $e^{\phi i} = \cos\phi + i\sin\phi$, so we want $\phi = \pi$. Therefore $\ln(-1) = \pi i$.