Is it the field of a particle?

We have a simple equation, based on Coulomb’s law, for the electric field surrounding a charged particle. Looking at figure n, we can imagine that if the current segment $d\ell$ was very short, then it might only contain one electron. It’s tempting, then, to interpret the Biot-Savart law as a similar equation for the magnetic field surrounding a moving charged particle. Tempting but wrong! Suppose you stand at a certain point in space and watch a charged particle move by. It has an electric field, and since it’s moving, you will also detect a magnetic field on top of that. Both of these fields change over time, however. Not only do they change their magnitudes and directions due to your changing geometric relationship to the particle, but they are also time-delayed, because disturbances in the electromagnetic field travel at the speed of light, which is finite. The fields you detect are the ones corresponding to where the particle used to be, not where it is now. Coulomb’s law and the Biot-Savart law are both false in this situation, since neither equation includes time as a variable. It’s valid to think of Coulomb’s law as the equation for the field of a stationary charged particle, but not a moving one. The Biot-Savart law fails completely as a description of the field of a charged particle, since stationary particles don’t make magnetic fields, and the Biot-Savart law fails in the case where the particle is moving.

If you look back at the long chain of reasoning that led to the Biot-Savart law, it all started from the relativistic arguments at the beginning of this chapter, where we assumed a steady current in an infinitely long wire. Everything that came later was built on this foundation, so all our reasoning depends on the assumption that the currents are steady. In a steady current, any charge that moves away from a certain spot is replaced by more charge coming up behind it, so even though the charges are all moving, the electric and magnetic fields they produce are constant. Problems of this type are called electrostatics and magnetostatics problems, and it is only for these problems that Coulomb’s law and the Biot-Savart law are valid.

You might think that we could patch up Coulomb’s law and the Biot-Savart law by inserting the appropriate time delays. However, we’ve already seen a clear example of a phenomenon that wouldn’t be fixed by this patch: on page 622, we found that a changing magnetic field creates an electric field. Induction effects like these also lead to the existence of light, which is a wave disturbance in the electric and magnetic fields. We could try to apply another band-aid fix to Coulomb’s law and the Biot-Savart law to make them deal with induction, but it won’t work.

So what are the fundamental equations that describe how sources give rise to electromagnetic fields? We’ve already encountered two of them: Gauss’ law for electricity and Gauss’ law for magnetism.
Experiments show that these are valid in all situations, not just static ones. But Gauss’ law for magnetism merely says that the magnetic flux through a closed surface is zero. It doesn’t tell us how to make magnetic fields using currents. It only tells us that we can’t make them using magnetic monopoles. The following section develops a new equation, called Ampère’s law, which is equivalent to the Biot-Savart law for magnetostatics, but which, unlike the Biot-Savart law, can easily be extended to nonstatic situations.

11.3 Magnetic fields by Ampère’s law

11.3.1 Ampère’s law

As discussed at the end of subsection 11.2.5, our goal now is to find an equation for magnetism that, unlike the Biot-Savart law, will not end up being a dead end when we try to extend it to nonstatic situations. Experiments show that Gauss’ law is valid in both static and nonstatic situations, so it would be reasonable to look for an approach to magnetism that is similar to the way Gauss’ law deals with electricity.

How can we do this? Figure a, reproduced from page 692, is our roadmap. Electric fields spread out from charges. Magnetic fields curl around currents. In figure b/1, we define a Gaussian surface, and we define the flux in terms of the electric field pointing out through this surface. In the magnetic case, b/2, we define a surface, called an Ampérian surface, and we define a quantity called the circulation, \( \Gamma \) (uppercase Greek gamma), in terms of the magnetic field that points along the edge of the Ampérian surface, c. We break the edge into tiny parts \( s_j \), and for each of these parts, we define a contribution to the circulation using the dot product of \( ds \) with the magnetic field:

\[
\Gamma = \sum s_j \cdot B_j
\]

The circulation is a measure of how curly the field is. Like a Gaussian surface, an Ampérian surface is purely a mathematical construction. It is not a physical object.

In figure b/2, the field is perpendicular to the edges on the ends, but parallel to the top and bottom edges. A dot product is zero when the vectors are perpendicular, so only the top and bottom edges contribute to \( \Gamma \). Let these edges have length \( s \). Since the field is constant along both of these edges, we don’t actually have to break them into tiny parts; we can just have \( s_1 \) on the top edge, pointing to the left, and \( s_2 \) on the bottom edge, pointing to the right. The vector \( s_1 \) is in the same direction as the field \( B_1 \), and \( s_2 \) is in the same direction as \( B_2 \), so the dot products are simply equal to

---

*If you didn’t read this optional subsection, don’t worry, because the point is that we need to try a whole new approach anyway.*
the products of the vectors’ magnitudes. The resulting circulation is

\[ \Gamma = |s_1||B_1| + |s_2||B_2| \]
\[ = \frac{2\pi k \eta s}{c^2} + \frac{2\pi k \eta s}{c^2} \]
\[ = \frac{4\pi k \eta s}{c^2}. \]

But \( \eta s \) is \((\text{current}/\text{length})(\text{length})\), i.e., it is the amount of current that pierces the Amp\’erian surface. We’ll call this current \( I_{\text{through}} \). We have found one specific example of the general law of nature known as Amp\’ere’s law:

\[ \Gamma = \frac{4\pi k}{c^2} I_{\text{through}}. \]

**Positive and negative signs**

Figures d/1 and d/2 show what happens to the circulation when we reverse the direction of the current \( I_{\text{through}} \). Reversing the current causes the magnetic field to reverse itself as well. The dot products occurring in the circulation are all negative in d/2, so the total circulation is now negative. To preserve Amp\’ere’s law, we need to define the current in d/2 as a negative number. In general, determine these plus and minus signs using the right-hand rule shown in the figure. As the fingers of your hand sweep around in the direction of the \( s \) vectors, your thumb defines the direction of current which is positive. Choosing the direction of the thumb is like choosing which way to insert an ammeter in a circuit: on a digital meter, reversing the connections gives readings which are opposite in sign.

**A solenoid example 13**

\[ \Gamma > 0, \ I_{\text{through}} > 0 \]
\[ \Gamma < 0, \ I_{\text{through}} < 0 \]

**d / Positive and negative signs in Amp\’ere’s law.**

- What is the field inside a long, straight solenoid of length \( \ell \) and radius \( a \), and having \( N \) loops of wire evenly wound along it, which carry a current \( I \)?

- This is an interesting example, because it allows us to get a very good approximation to the field, but without some experimental input it wouldn’t be obvious what approximation to use. Figure e/1 shows what we’d observe by measuring the field of a real solenoid. The field is nearly constant inside the tube, as long as we stay far away from the mouths. The field outside is much weaker. For the sake of an approximate calculation, we can idealize this field as shown in figure e/2. Of the edges of the Amp\’erian surface shown in e/3, only AB contributes to the flux — there is zero field along CD, and the field is perpendicular to edges BC

**e / Example 13: a cutaway view of a solenoid.**
A proof of Ampère's law.

and DA. Ampère's law gives

\[ \Gamma = \frac{4\pi k}{c^2} I_{\text{through}} \]
\[ (B)(\text{length of AB}) = \frac{4\pi k}{c^2} (\eta)(\text{length of AB}) \]
\[ B = \frac{4\pi k\eta}{c^2} \]
\[ = \frac{4\pi kNI}{c^2\ell} \]

**self-check D**
What direction is the current in figure e?  

**Answer, p. 1064**

**self-check E**
Based on how \( \ell \) entered into the derivation in example 13, how should it be interpreted? Is it the total length of the wire?  

**Answer, p. 1065**

**self-check F**
Surprisingly, we never needed to know the radius of the solenoid in example 13. Why is it physically plausible that the answer would be independent of the radius?  

**Answer, p. 1065**

Example 13 shows how much easier it can sometimes be to calculate a field using Ampère’s law rather than the approaches developed previously in this chapter. However, if we hadn’t already known something about the field, we wouldn’t have been able to get started. In situations that lack symmetry, Ampère’s law may make things harder, not easier. Anyhow, we will have no choice in nonstatic cases, where Ampère’s law is true, and static equations like the Biot-Savart law are false.

### 11.3.2 A quick and dirty proof

Here’s an informal sketch for a proof of Ampère’s law, with no pretensions to rigor. Even if you don’t care much for proofs, it would be a good idea to read it, because it will help to build your ability to visualize how Ampère’s law works.

First we establish by a direct computation (homework problem 26) that Ampère’s law holds for the geometry shown in figure f/1, a circular Ampérien surface with a wire passing perpendicularly through its center. If we then alter the surface as in figure f/2, Ampère’s law still works, because the straight segments, being perpendicular to the field, don’t contribute to the circulation, and the new arc makes the same contribution to the circulation as the old one it replaced, because the weaker field is compensated for by the greater length of the arc. It is clear that by a series of such modifications, we could mold the surface into any shape, f/3.

Next we prove Ampère’s law in the case shown in figure f/4: a small, square Ampérien surface subject to the field of a distant
square dipole. This part of the proof can be most easily accomplished by the methods of section 11.4. It should, for example, be plausible in the case illustrated here. The field on the left edge is stronger than the field on the right, so the overall contribution of these two edges to the circulation is slightly counterclockwise. However, the field is not quite perpendicular to the top and bottoms edges, so they both make small clockwise contributions. The clockwise and counterclockwise parts of the circulation end up canceling each other out. Once Ampère’s law is established for a square surface like \( f/4 \), it follows that it is true for an irregular surface like \( f/5 \), since we can build such a shape out of squares, and the circulations are additive when we paste the surfaces together this way.

By pasting a square dipole onto the wire, \( f/6 \), like a flag attached to a flagpole, we can cancel out a segment of the wire’s current and create a detour. Ampère’s law is still true because, as shown in the last step, the square dipole makes zero contribution to the circulation. We can make as many detours as we like in this manner, thereby morphing the wire into an arbitrary shape like \( f/7 \).

What about a wire like \( f/8 \)? It doesn’t pierce the Ampèrian surface, so it doesn’t add anything to \( I_{\text{through}} \), and we need to show that it likewise doesn’t change the circulation. This wire, however, could be built by tiling the half-plane on its right with square dipoles, and we’ve already established that the field of a distant dipole doesn’t contribute to the circulation. (Note that we couldn’t have done this with a wire like \( f/7 \), because some of the dipoles would have been right on top of the Ampèrian surface.)

If Ampère’s law holds for cases like \( f/7 \) and \( f/8 \), then it holds for any complex bundle of wires, including some that pass through the Ampèrian surface and some that don’t. But we can build just about any static current distribution we like using such a bundle of wires, so it follows that Ampère’s law is valid for any static current distribution.

### 11.3.3 Maxwell’s equations for static fields

Static electric fields don’t curl the way magnetic fields do, so we can state a version of Ampère’s law for electric fields, which is that the circulation of the electric field is zero. Summarizing what we know so far about static fields, we have

\[
\Phi_E = 4\pi k q_{\text{in}} \\
\Phi_B = 0 \\
\Gamma_E = 0 \\
\Gamma_B = \frac{4\pi k}{c^2} I_{\text{through}}.
\]

This set of equations is the static case of the more general relations known as Maxwell’s equations. On the left side of each equation, we
have information about a field. On the right is information about the field’s sources.

It is vitally important to realize that these equations are only true for statics. They are incorrect if the distribution of charges or currents is changing over time. For example, we saw on page 622 that the changing magnetic field in an inductor gives rise to an electric field. Such an effect is completely inconsistent with the static version of Maxwell’s equations; the equations don’t even refer to time, so if the magnetic field is changing over time, they will not do anything special. The extension of Maxwell’s equations to nonstatic fields is discussed in section 11.6.

**Discussion Questions**

A. Figure g/1 shows a wire with a circular Ampèrian surface drawn around its waist; in this situation, Ampère’s law can be verified easily based on the equation for the field of a wire. In panel 2, a second wire has been added. Explain why it’s plausible that Ampère’s law still holds.

B. Figure h is like figure g, but now the second wire is perpendicular to the first, and lies in the plane of, and outside of, the Ampèrian surface. Carry out a similar analysis.

C. This discussion question is similar to questions A and B, but now the Ampèrian surface has been moved off center.

D. The left-hand wire has been nudged over a little. Analyze as before.

E. You know what to do.
### 11.4 Ampère’s law in differential form (optional)

#### 11.4.1 The curl operator

The differential form of Gauss’ law is more physically satisfying than the integral form, because it relates the charges that are present at some point to the properties of the electric field at the same point. Likewise, it would be more attractive to have a differential version of Ampère’s law that would relate the currents to the magnetic field at a single point. Intuitively, the divergence was based on the idea of the div-meter, a/1. The corresponding device for measuring the curliness of a field is the curl-meter, a/2. If the field is curly, then the torques on the charges will not cancel out, and the wheel will twist against the resistance of the spring. If your intuition tells you that the curl-meter will never do anything at all, then your intuition is doing a beautiful job on static fields; for nonstatic fields, however, it is perfectly possible to get a curly electric field.

Gauss’ law in differential form relates $\text{div } E$, a scalar, to the charge density, another scalar. Ampère’s law, however, deals with directions in space: if we reverse the directions of the currents, it makes a difference. We therefore expect that the differential form of Ampère’s law will have vectors on both sides of the equal sign, and we should be thinking of the curl-meter’s result as a vector. First we find the orientation of the curl-meter that gives the strongest torque, and then we define the direction of the curl vector using the right-hand rule shown in figure a/3.

To convert the div-meter concept to a mathematical definition, we found the infinitesimal flux, $d\Phi$ through a tiny cubical Gaussian surface containing a volume $dV$. By analogy, we imagine a tiny square Ampérien surface with infinitesimal area $dA$. We assume this surface has been oriented in order to get the maximum circulation. The area vector $dA$ will then be in the same direction as the one defined in figure a/3. Ampère’s law is

$$d\Gamma = \frac{4\pi k}{c^2} dI_{\text{through}}.$$ 

We define a current density per unit area, $j$, which is a vector pointing in the direction of the current and having magnitude $j = dI/|dA|$. In terms of this quantity, we have

$$d\Gamma = \frac{4\pi k}{c^2} j |j| dA$$

$$\frac{d\Gamma}{|dA|} = \frac{4\pi k}{c^2} |j|$$

With this motivation, we define the magnitude of the curl as

$$|\text{curl } B| = \frac{d\Gamma}{|dA|}.$$
The coordinate system used in the following examples.

The field $\hat{x}$.

The field $\hat{y}$.

The field $x\hat{y}$.

The curl, just like a derivative, has a differential divided by another differential. In terms of this definition, we find Ampère’s law in differential form:

$$\text{curl } B = \frac{4\pi k}{c^2} j$$

The complete set of Maxwell’s equations in differential form is collected on page 1029.

11.4.2 Properties of the curl operator

The curl is a derivative.

As an example, let’s calculate the curl of the field $\hat{x}$ shown in figure c. For our present purposes, it doesn’t really matter whether this is an electric or a magnetic field; we’re just getting out feet wet with the curl as a mathematical definition. Applying the definition of the curl directly, we construct an Ampérian surface in the shape of an infinitesimally small square. Actually, since the field is uniform, it doesn’t even matter very much whether we make the square finite or infinitesimal. The right and left edges don’t contribute to the circulation, since the field is perpendicular to these edges. The top and bottom do contribute, but the top’s contribution is clockwise, i.e., into the page according to the right-hand rule, while the bottom contributes an equal amount in the counterclockwise direction, which corresponds to an out-of-the-page contribution to the curl. They cancel, and the circulation is zero. We could also have determined this by imagining a curl-meter inserted in this field: the torques on it would have canceled out.

It makes sense that the curl of a constant field is zero, because the curl is a kind of derivative. The derivative of a constant is zero.

The curl is rotationally invariant.

Figure c looks just like figure c, but rotated by 90 degrees. Physically, we could be viewing the same field from a point of view that was rotated. Since the laws of physics are the same regardless of rotation, the curl must be zero here as well. In other words, the curl is rotationally invariant. If a certain field has a certain curl vector, then viewed from some other angle, we simply see the same field and the same curl vector, viewed from a different angle. A zero vector viewed from a different angle is still a zero vector.

As a less trivial example, let’s compute the curl of the field $F = x\hat{y}$ shown in figure e, at the point $(x = 0, y = 0)$. The circulation around a square of side $s$ centered on the origin can be approximated
by evaluating the field at the midpoints of its sides,

\[
\begin{align*}
  x &= s/2 & y &= 0 & \mathbf{F} &= (s/2)\hat{y} & s_1 \cdot \mathbf{F} &= s^2/2 \\
  x &= 0 & y &= s/2 & \mathbf{F} &= 0 & s_2 \cdot \mathbf{F} &= 0 \\
  x &= -s/2 & y &= 0 & \mathbf{F} &= -(s/2)\hat{y} & s_3 \cdot \mathbf{F} &= s^2/2 \\
  x &= 0 & y &= -s/2 & \mathbf{F} &= 0 & s_4 \cdot \mathbf{F} &= 0,
\end{align*}
\]

which gives a circulation of \(s^2\), and a curl with a magnitude of \(s^2/\text{area} = s^2/s^2 = 1\). By the right-hand rule, the curl points out of the page, i.e., along the positive \(z\) axis, so we have

\[
\text{curl } x\hat{y} = \hat{z}.
\]

Now consider the field \(-y\hat{x}\), shown in figure f. This is the same as the previous field, but with your book rotated by 90 degrees about the \(z\) axis. Rotating the result of the first calculation, \(\hat{z}\), about the \(z\) axis doesn’t change it, so the curl of this field is also \(\hat{z}\).

**Scaling**

When you’re taking an ordinary derivative, you have the rule

\[
\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x).
\]

In other words, multiplying a function by a constant results in a derivative that is multiplied by that constant. The curl is a kind of derivative operator, and the same is true for a curl.

\textit{Multiplying the field by \(-1\).} \textit{example 14}

\begin{itemize}
  \item What is the curl of the field \(-x\hat{y}\) at the origin?
  \item Using the scaling property just discussed, we can make this into a curl that we’ve already calculated:
\end{itemize}

\[
\text{curl } (-x\hat{y}) = -\text{curl } (x\hat{y}) = -\hat{z}
\]

This is in agreement with the right-hand rule.

**The curl is additive.**

We have only calculated each field’s curl at the origin, but each of these fields actually has the same curl everywhere. In example 14, for instance, it is obvious that the curl is constant along any vertical line. But even if we move along the \(x\) axis, there is still an imbalance between the torques on the left and right sides of the curlmeter. More formally, suppose we start from the origin and move to the left by one unit. We find ourselves in a region where the field
is very much as it was before, except that all the field vectors have had one unit worth of \( \mathbf{\hat{y}} \) added to them. But what do we get if we take the curl of \(-x\mathbf{\hat{y}} + \mathbf{\hat{y}}\)? The curl, like any god-fearing derivative operation, has the additive property

\[
\text{curl } (\mathbf{F} + \mathbf{G}) = \text{curl } \mathbf{F} + \text{curl } \mathbf{G},
\]

so

\[
\text{curl } (-x\mathbf{\hat{y}} + \mathbf{\hat{y}}) = \text{curl } (-x\mathbf{\hat{y}}) + \text{curl } (\mathbf{\hat{y}}).
\]

But the second term is zero, so we get the same result as at the origin.

A field that goes in a circle example 15
▷ What is the curl of the field \( x\mathbf{\hat{y}} - y\mathbf{\hat{x}} \)?

▷ Using the linearity of the curl, and recognizing each of the terms as one whose curl we have already computed, we find that this field’s curl is a constant \(2\mathbf{\hat{z}}\). This agrees with the right-hand rule.

The field inside a long, straight wire example 16
▷ What is the magnetic field inside a long, straight wire in which the current density is \( \mathbf{j} \)?

▷ Let the wire be along the \( z \) axis, so \( \mathbf{j} = j\mathbf{\hat{z}} \). Ampère’s law gives

\[
\text{curl } \mathbf{B} = \frac{4\pi k}{c^2} j\mathbf{\hat{z}}.
\]

In other words, we need a magnetic field whose curl is a constant. We’ve encountered several fields with constant curls, but the only one that has the same symmetry as the cylindrical wire is \( x\mathbf{\hat{y}} - y\mathbf{\hat{x}} \), so the answer must be this field or some constant multiplied by it,

\[
\mathbf{B} = b(x\mathbf{\hat{y}} - y\mathbf{\hat{x}}).
\]

The curl of this field is \(2b\mathbf{\hat{z}}\), so

\[
2b = \frac{4\pi k}{c^2} j,
\]

and thus

\[
\mathbf{B} = \frac{2\pi k}{c^2} j(x\mathbf{\hat{y}} - y\mathbf{\hat{x}}).
\]

The curl in component form

Now consider the field

\[
F_x = ax + by + c
\]

\[
F_y = dx + ey + f,
\]
i.e.,

\[ \mathbf{F} = ax\mathbf{\hat{x}} + by\mathbf{\hat{y}} + cx\mathbf{\hat{z}} + dxy\mathbf{\hat{y}} + ey\mathbf{\hat{y}} + f\mathbf{\hat{y}}. \]

The only terms whose curls we haven’t yet explicitly computed are the \( a, e, \) and \( f \) terms, and their curls turn out to be zero (homework problem 50). Only the \( b \) and \( d \) terms have nonvanishing curls. The curl of this field is

\[
\text{curl } \mathbf{F} = \text{curl } (by\mathbf{\hat{x}}) + \text{curl } (dx\mathbf{\hat{y}})
\]

\[
= b\text{curl } (y\mathbf{\hat{x}}) + d\text{curl } (x\mathbf{\hat{y}}) \quad [\text{scaling}]
\]

\[
= b(-\mathbf{\hat{z}}) + d(\mathbf{\hat{z}}) \quad [\text{found previously}]
\]

\[
= (d - b)\mathbf{\hat{z}}.
\]

But \emph{any} field in the \( x - y \) plane can be approximated with this type of field, as long as we only need to get a good approximation within a small region. The infinitesimal Ampèrean surface occurring in the definition of the curl is tiny enough to fit in a pretty small region, so we can get away with this here. The \( d \) and \( b \) coefficients can then be associated with the partial derivatives \( \partial F_y/\partial x \) and \( \partial F_x/\partial y \). We therefore have

\[
\text{curl } \mathbf{F} = \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{\hat{z}}
\]

for any field in the \( x - y \) plane. In three dimensions, we just need to generate two more equations like this by doing a cyclic permutation of the variables \( x, y, \) and \( z \):

\[
\begin{align*}
\text{(curl } \mathbf{F})_x &= \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\
\text{(curl } \mathbf{F})_y &= \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\
\text{(curl } \mathbf{F})_z &= \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}
\end{align*}
\]

\textbf{A sine wave example 17}

\( \triangleright \) Find the curl of the following electric field

\[ \mathbf{E} = (\sin x)\mathbf{\hat{y}}, \]

and interpret the result.

\( \triangleright \) The only nonvanishing partial derivative occurring in this curl is

\[
\text{(curl } \mathbf{E})_x = \frac{\partial E_y}{\partial x} = \cos x,
\]
This is visually reasonable: the curl-meter would spin if we put its wheel in the plane of the page, with its axle poking out along the z axis. In some areas it would spin clockwise, in others counterclockwise, and this makes sense, because the cosine is positive in some placed and negative in others.

This is a perfectly reasonable field pattern: it the electric field pattern of a light wave! But Ampère’s law for electric fields says the curl of \( \mathbf{E} \) is supposed to be zero. What’s going on? What’s wrong is that we can't assume the static version of Ampère’s law. All we’ve really proved is that this pattern is impossible as a static field: we can’t have a light wave that stands still.

Figure k is a summary of the vector calculus presented in the optional sections of this book. The first column shows that one function is related to another by a kind of differentiation. The second column states the fundamental theorem of calculus, which says that if you integrate the derivative over the interior of a region, you get some information about the original function at the boundary of that region.
11.5 Induced electric fields

11.5.1 Faraday’s experiment

Nature is simple, but the simplicity may not become evident until a hundred years after the discovery of some new piece of physics. We’ve already seen, on page 622, that the time-varying magnetic field in an inductor causes an electric field. This electric field is not created by charges. That argument, however, only seems clear with hindsight. The discovery of this phenomenon of induced electric fields — fields that are not due to charges — was a purely experimental accomplishment by Michael Faraday (1791-1867), the son of a blacksmith who had to struggle against the rigid class struc-
Faraday on a British banknote.

Figure b is a simplified drawing of the following experiment, as described in Faraday’s original paper: “Two hundred and three feet of copper wire . . . were passed round a large block of wood; [another] two hundred and three feet of similar wire were interposed as a spiral between the turns of the first, and metallic contact everywhere prevented by twine [insulation]. One of these [coils] was connected with a galvanometer [voltmeter], and the other with a battery. . . When the contact was made, there was a sudden and very slight effect at the galvanometer, and there was also a similar slight effect when the contact with the battery was broken. But whilst the . . . current was continuing to pass through the one [coil], no . . . effect . . . upon the other [coil] could be perceived, although the active power of the battery was proved to be great, by its heating the whole of its own coil [through ordinary resistive heating] . . .”

From Faraday’s notes and publications, it appears that the situation in figure b/3 was a surprise to him, and he probably thought it would be a surprise to his readers, as well. That’s why he offered evidence that the current was still flowing: to show that the battery hadn’t just died. The induction effect occurred during the short time it took for the black coil’s magnetic field to be established, b/2. Even more counterintuitively, we get an effect, equally strong but in the opposite direction, when the circuit is broken, b/4. The effect occurs only when the magnetic field is changing, and it appears to be proportional to the derivative $\frac{\partial B}{\partial t}$, which is in one direction when the field is being established, and in the opposite direction when it collapses.

The effect is proportional to $\frac{\partial B}{\partial t}$, but what *is* the effect? A voltmeter is nothing more than a resistor with an attachment for measuring the current through it. A current will not flow through a resistor unless there is some electric field pushing the electrons, so we conclude that the changing magnetic field has produced an electric field in the surrounding space. Since the white wire is not a perfect conductor, there must be electric fields in it as well. The remarkable thing about the circuit formed by the white wire is that as the electrons travel around and around, they are always being pushed forward by electric fields. This violates the loop rule, which says that when an electron makes a round trip, there is supposed to be just as much “uphill” (moving against the electric field) as “downhill” (moving with it). That’s OK. The loop rule is only true for statics. Faraday’s experiments show that an electron really can go around and around, and always be going “downhill,” as in the famous drawing by M.C. Escher shown in figure c. That’s just what happens when you have a curly field.
When a field is curly, we can measure its curliness using a circulation. Unlike the magnetic circulation $\Gamma_B$, the electric circulation $\Gamma_E$ is something we can measure directly using ordinary tools. A circulation is defined by breaking up a loop into tiny segments, $ds$, and adding up the dot products of these distance vectors with the field. But when we multiply electric field by distance, what we get is an indication of the amount of work per unit charge done on a test charge that has been moved through that distance. The work per unit charge has units of volts, and it can be measured using a voltmeter, as shown in figure e, where $\Gamma_E$ equals the sum of the voltmeter readings. Since the electric circulation is directly measurable, most people who work with circuits are more familiar with it than they are with the magnetic circulation. They usually refer to $\Gamma_E$ using the synonym “emf,” which stands for “electromotive force,” and notate it as $\mathcal{E}$. (This is an unfortunate piece of terminology, because its units are really volts, not newtons.) The term emf can also be used when the path is not a closed loop.

Faraday’s experiment demonstrates a new relationship

$$\Gamma_E \propto -\frac{\partial B}{\partial t},$$

where the negative sign is a way of showing the observed left-handed relationship, d. This is similar to the structure of of Ampère’s law:

$$\Gamma_B \propto I_{\text{through}},$$

which also relates the curliness of a field to something that is going on nearby (a current, in this case).

It’s important to note that even though the emf, $\Gamma_E$, has units of volts, it isn’t a voltage. A voltage is a measure of the electrical energy a charge has when it is at a certain point in space. The curly nature of nonstatic fields means that this whole concept becomes nonsense. In a curly field, suppose one electron stays at home while its friend goes for a drive around the block. When they are reunited, the one that went around the block has picked up some kinetic energy, while the one who stayed at home hasn’t. We simply can’t define an electrical energy $U_e = qV$ so that $U_e + K$ stays the same for each electron. No voltage pattern, $V$, can do this, because then it would predict the same kinetic energies for the two electrons, which is incorrect. When we’re dealing with nonstatic fields, we need to think of the electrical energy in terms of the energy density of the fields themselves.

It might sound as though an electron could get a free lunch by circling around and around in a curly electric field, resulting in a violation of conservation of energy. The following examples, in addition to their practical interest, both show that energy is in fact conserved.
The generator  

A basic generator, f, consists of a permanent magnet that rotates within a coil of wire. The magnet is turned by a motor or crank, (not shown). As it spins, the nearby magnetic field changes. This changing magnetic field results in an electric field, which has a curly pattern. This electric field pattern creates a current that whips around the coils of wire, and we can tap this current to light the lightbulb.

If the magnet was on a frictionless bearing, could we light the bulb for free indefinitely, thus violating conservation of energy? No. Mechanical work has to be done to crank the magnet, and that’s where the energy comes from. If we break the light-bulb circuit, it suddenly gets easier to crank the magnet! This is because the current in the coil sets up its own magnetic field, and that field exerts a torque on the magnet. If we stopped cranking, this torque would quickly make the magnet stop turning.

Self-check G

When you’re driving your car, the engine recharges the battery continuously using a device called an alternator, which is really just a generator. Why can’t you use the alternator to start the engine if your car’s battery is dead?

The transformer  

In example 18 on page 562, we discussed the advantages of transmitting power over electrical lines using high voltages and low currents. However, we don’t want our wall sockets to operate at 10000 volts! For this reason, the electric company uses a device called a transformer, g, to convert everything to lower voltages and higher currents inside your house. The coil on the input side creates a magnetic field. Transformers work with alternating current, so the magnetic field surrounding the input coil is always changing. This induces an electric field, which drives a current around the output coil.

Since the electric field is curly, an electron can keep gaining more and more energy by circling through it again and again. Thus the output voltage can be controlled by changing the number of coils of wire on the output side. Changing the number of coils on the input side also has an effect (homework problem 33).

In any case, conservation of energy guarantees that the amount of power on the output side must equal the amount put in originally, \( I_{in} V_{in} = I_{out} V_{out} \), so no matter what factor the voltage is reduced by, the current is increased by the same factor.

Discussion Questions

A Suppose the bar magnet in figure f on page 716 has a magnetic field pattern that emerges from its top, circling around and coming back in the bottom. This field is created by electrons orbiting atoms inside
the magnet. Are these atomic currents clockwise or counterclockwise as seen from above? In what direction is the current flowing in the circuit?

We have a circling atomic current inside the circling current in the wires. When we have two circling currents like this, they will make torques on each other that will tend to align them in a certain way. Since currents in the same direction attract one another, which way is the torque made by the wires on the bar magnet? Verify that due to this torque, mechanical work has to be done in order to crank the generator.

11.5.2 Why induction?

Faraday’s results leave us in the dark about several things:

- They don’t explain why induction effects occur.
- The relationship $\Gamma_E \propto -\partial B / \partial t$ tells us that a changing magnetic field creates an electric field in the surrounding region of space, but the phrase “surrounding region of space” is vague, and needs to be made mathematical.
- Suppose that we can make the “surrounding region of space” idea more well defined. We would then want to know the proportionality constant that has been hidden by the $\propto$ symbol. Although experiments like Faraday’s could be used to find a numerical value for this constant, we would like to know why it should have that particular value.

We can get some guidance from the example of a car’s alternator (which just means generator), referred to in the self-check on page 716. To keep things conceptually simple, I carefully avoided mentioning that in a real car’s alternator, it isn’t actually the permanent magnet that spins. The coil is what spins. The choice of design h/1 or h/2 is merely a matter of engineering convenience, not physics. All that matters is the relative motion of the two objects.

This is highly suggestive. As discussed at the beginning of this chapter, magnetism is a relativistic effect. From arguments about relative motion, we concluded that moving electric charges create magnetic fields. Now perhaps we can use reasoning with the same flavor to show that changing magnetic fields produce curly electric fields. Note that figure h/2 doesn’t even require induction. The protons and electrons in the coil are moving through a magnetic field, so they experience forces. The protons can’t flow, because the coil is a solid substance, but the electrons can, so a current is induced.7

Now if we’re convinced that figure h/2 produces a current in the coil, then it seems very plausible that the same will happen in

7 Note that the magnetic field never does work on a charged particle, because its force is perpendicular to the motion; the electric power is actually coming from the mechanical work that had to be done to spin the coil. Spinning the coil is more difficult due to the presence of the magnet.
A generator that works with linear motion.

Figure h/1, which implies the existence of induction effects. But this example involves circular motion, so it doesn’t quite work as a way of proving that induction exists. When we say that motion is relative, we only mean straight-line motion, not circular motion.

A more ironclad relativistic argument comes from the arrangement shown in figure i. This is also a generator — one that is impractical, but much easier to understand.

Flea 1 doesn’t believe in this modern foolishness about induction. She’s sitting on the bar magnet, which to her is obviously at rest. As the square wire loop is dragged away from her and the magnet, its protons experience a force out of the page, because the cross product $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ is out of the page. The electrons, which are negatively charged, feel a force into the page. The conduction electrons are free to move, but the protons aren’t. In the front and back sides of the loop, this force is perpendicular to the wire. In the right and left sides, however, the electrons are free to respond to the force. Note that the magnetic field is weaker on the right side. It’s as though we had two pumps in a loop of pipe, with the weaker pump trying to push in the opposite direction; the weaker pump loses the argument. We get a current that circulates around the loop.

There is no induction going on in this frame of reference; the forces that cause the current are just the ordinary magnetic forces experienced by any charged particle moving through a magnetic field.

Flea 2 is sitting on the loop, which she considers to be at rest. In her frame of reference, it’s the bar magnet that is moving. Like flea 1, she observes a current circulating around the loop, but unlike flea 1, she cannot use magnetic forces to explain this current. As far as she is concerned, the electrons were initially at rest. Magnetic forces are forces between moving charges and other moving charges, so a magnetic field can never accelerate a charged particle starting from rest. A force that accelerates a charge from rest can only be an electric force, so she is forced to conclude that there is an electric field in her region of space. This field drives electrons around and around in circles, so it is apparently violating the loop rule — it is a curly field. What reason can flea 2 offer for the existence of this electric field pattern? Well, she’s been noticing that the magnetic

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8If the pump analogy makes you uneasy, consider what would happen if all the electrons moved into the page on both sides of the loop. We’d end up with a net negative charge at the back side, and a net positive charge on the front. This actually would happen in the first nanosecond after the loop was set in motion. This buildup of charge would start to quench both currents due to electrical forces, but the current in the right side of the wire, which is driven by the weaker magnetic field, would be the first to stop. Eventually, an equilibrium will be reached in which the same amount of current is flowing at every point around the loop, and no more charge is being piled up.

9The wire is not a perfect conductor, so this current produces heat. The energy required to produce this heat comes from the hands, which are doing mechanical work as they separate the magnet from the loop.
field in her region of space has been changing, possibly because that bar magnet over there has been getting farther away. She observes that a changing magnetic field creates a curly electric field.

We therefore conclude that induction effects must exist based on the fact that motion is relative. If we didn’t want to admit induction effects, we would have to outlaw flea 2’s frame of reference, but the whole idea of relative motion is that all frames of reference are created equal, and there is no way to determine which one is really at rest.

This whole line of reasoning was not available to Faraday and his contemporaries, since they thought the relative nature of motion only applied to matter, not to electric and magnetic fields. But with the advantage of modern hindsight, we can understand in fundamental terms the facts that Faraday had to take simply as mysterious experimental observations. For example, the geometric relationship shown in figure d follows directly from the direction of the current we deduced in the story of the two fleas.

11.5.3 Faraday’s law

We can also answer the other questions posed on page 717. The divide-and-conquer approach should be familiar by now. We first determine the circulation $\Gamma_E$ in the case where the wire loop is very tiny, j. Then we can break down any big loop into a grid of small ones; we’ve already seen that when we make this kind of grid, the circulations add together. Although we’ll continue to talk about a physical loop of wire, as in figure i, the tiny loop can really be just like the edges of an Ampérian surface: a mathematical construct that doesn’t necessarily correspond to a real object.

In the close-up view shown in figure j, the field looks simpler. Just as a tiny part of a curve looks straight, a tiny part of this magnetic field looks like the field vectors are just getting shorter by the same amount with each step to the right. Writing $dx$ for the width of the loop, we therefore have

$$B(x + dx) - B(x) = \frac{\partial B}{\partial x} dx$$

for the difference in the strength of the field between the left and right sides. In the frame of reference where the loop is moving, a charge $q$ moving along with the loop at velocity $v$ will experience a magnetic force $F_B = qvB\hat{y}$. In the frame moving along with the loop, this is interpreted as an electrical force, $F_E = qE\hat{y}$. Observers in the two frames agree on how much force there is, so in the loop’s frame, we have an electric field $E = vB\hat{y}$. This field is

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10They can’t be blamed too much for this. As a consequence of Faraday’s work, it soon became apparent that light was an electromagnetic wave, and to reconcile this with the relative nature of motion requires Einstein’s version of relativity, with all its subversive ideas how space and time are not absolute.
perpendicular to the front and back sides of the loop, BC and DA, so there is no contribution to the circulation along these sides, but there is a counterclockwise contribution to the circulation on CD, and smaller clockwise one on AB. The result is a circulation that is counterclockwise, and has an absolute value

$$|\Gamma_E| = |E(x)\, dy - E(x + dx)\, dy|$$

$$= |v[B(x) - B(x + dx)]|\, dy$$

$$= \left| v \frac{\partial B}{\partial x} \right| dx\, dy$$

$$= \left| \frac{dx}{dt} \frac{\partial B}{\partial x} \right| dx\, dy$$

$$= \left| \frac{\partial B}{\partial t} \right| dA.$$

Using a right-hand rule, the counterclockwise circulation is represented by pointing one's thumb up, but the vector $\frac{\partial B}{\partial t}$ is down. This is just a rephrasing of the geometric relationship shown in figure d on page 715. We can represent the opposing directions using a minus sign,

$$\Gamma_E = -\frac{\partial B}{\partial t} \, dA.$$

Although this derivation was carried out with everything aligned in a specific way along the coordinate axes, it turns out that this relationship can be generalized as a vector dot product,

$$\Gamma_E = -\frac{\partial B}{\partial t} \cdot dA.$$

Finally, we can take a finite-sized loop and break down the circulation around its edges into a grid of tiny loops. The circulations add, so we have

$$\Gamma_E = -\sum \frac{\partial B_j}{\partial t} \cdot dA_j.$$

This is known as Faraday’s law. (I don’t recommend memorizing all these names.) Mathematically, Faraday’s law is very similar to the structure of Ampère’s law: the circulation of a field around the edges of a surface is equal to the sum of something that points through the

If the loop itself isn’t moving, twisting, or changing shape, then the area vectors don’t change over time, and we can move the derivative outside the sum, and rewrite Faraday’s law in a slightly more transparent form:

$$\Gamma_E = -\frac{\partial}{\partial t} \sum \frac{\partial B_j}{\partial t} \cdot dA_j$$

$$= \frac{\partial \Phi_B}{\partial t}.$$
A changing magnetic flux makes a curly electric field. You might think based on Gauss’ law for magnetic fields that $\Phi_B$ would be identically zero. However, Gauss’ law only applies to surfaces that are closed, i.e., have no edges.

**self-check H**
Check that the units in Faraday’s law work out. An easy way to approach this is to use the fact that $vB$ has the same units as $E$, which can be seen by comparing the equations for magnetic and electric forces used above.

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**A pathetic generator example 20**

The horizontal component of the earth’s magnetic field varies from zero, at a magnetic pole, to about $10^{-4}$ T near the equator. Since the distance from the equator to a pole is about $10^7$ m, we can estimate, very roughly, that the horizontal component of the earth’s magnetic field typically varies by about $10^{-11}$ T/m as you go north or south. Suppose you connect the terminals of a one-ohm lightbulb to each other with a loop of wire having an area of 1 m$^2$. Holding the loop so that it lies in the east-west-up-down plane, you run straight north at a speed of 10 m/s, how much current will flow? Next, repeat the same calculation for the surface of a neutron star. The magnetic field on a neutron star is typically $10^9$ T, and the radius of an average neutron star is about $10^4$ m.

Let’s work in the frame of reference of the running person. In this frame of reference, the earth is moving, and therefore the local magnetic field is changing in strength by $10^{-9}$ T/s. This rate of change is almost exactly the same throughout the interior of the loop, so we can dispense with the summation, and simply write Faraday’s law as

$$\Gamma_E = -\frac{\partial B}{\partial t} \cdot A.$$  

Since what we estimated was the rate of change of the horizontal component, and the vector $A$ is horizontal (perpendicular to the loop), we can find this dot product simply by multiplying the two numbers:

$$\Gamma_E = (10^{-9} \text{ T/s})(1 \text{ m}^2)$$
$$= 10^{-9} \text{ T} \cdot \text{m}^2 / \text{s}$$
$$= 10^{-9} \text{ V}$$

This is certainly not enough to light the bulb, and would not even be easy to measure using the most sensitive laboratory instruments.

Now what about the neutron star? We’ll pretend you’re tough enough that its gravity doesn’t instantly crush you. The spatial
variation of the magnetic field is on the order of \((10^9 \text{T}/10^4 \text{m}) = 10^5 \text{T/m}\). If you can run north at the same speed of 10 m/s, then in your frame of reference there is a temporal (time) variation of about \(10^6 \text{T/s}\), and a calculation similar to the previous one results in an emf of \(10^6 \text{V}\). This isn’t just strong enough to light the bulb, it’s sufficient to evaporate it, and kill you as well!

It might seem as though having access to a region of rapidly changing magnetic field would therefore give us an infinite supply of free energy. However, the energy that lights the bulb is actually coming from the mechanical work you do by running through the field. A tremendous force would be required to make the wire loop move through the neutron star’s field at any significant speed.

\[ \text{Speed and power in a generator example 21} \]

\(\text{Figure } k\) shows three graphs of the magnetic flux through a generator’s coils as a function of time. In graph 2, the generator is being cranked at twice the frequency. In 3, a permanent magnet with double the strength has been used. In 4, the generator is being cranked in the opposite direction. Compare the power generated in figures 2-4 with the the original case, 1.

\(\text{If the flux varies as } \Phi = A \sin \omega t, \text{ then the time derivative occurring in Faraday’s law is } \partial \Phi / \partial t = A \omega \cos \omega t. \text{ The absolute value of this is the same as the absolute value of the emf, } \Gamma_E. \text{ The current through the lightbulb is proportional to this emf, and the power dissipated depends on the square of the current (} P = i^2 R\text{), so } P \propto A^2 \omega^2. \text{ Figures 2 and 3 both give four times the output power (and require four times the input power). Figure 4 gives the same result as figure 1; we can think of this as a negative amplitude, which gives the same result when squared.} \]

\[ \text{An approximate loop rule example 22} \]

\(\text{Figure } l/1\) shows a simple RL circuit of the type discussed in the last chapter. A current has already been established in the coil, let’s say by a battery. The battery was then unclipped from the coil, and we now see the circuit as the magnetic field in and around the inductor is beginning to collapse. I’ve already cautioned you that the loop rule doesn’t apply in nonstatic situations, so we can’t assume that the readings on the four voltmeters add up to zero. The interesting thing is that although they don’t add up to exactly zero in this circuit, they very nearly do. Why is the loop rule even approximately valid in this situation?

The reason is that the voltmeters are measuring the emf \(\Gamma_E\) around the path shown in figure \(l/2\), and the stray field of the solenoid is extremely weak out there. In the region where the meters are, the arrows representing the magnetic field would be too small to allow me to draw them to scale, so I have simply omitted them. Since the field is so weak in this region, the flux through the loop is nearly zero, and the rate of change of the flux, \(\partial \Phi_B / \partial t\), is also
nearly zero. By Faraday's law, then, the emf around this loop is nearly zero.

Now consider figure 1/3. The flux through the interior of this path is not zero, because the strong part of the field passes through it, and not just once but many times. To visualize this, imagine that we make a wire frame in this shape, dip it in a tank of soapy water, and pull it out, so that there is a soap-bubble film spanning its interior. Faraday's law refers to the rate of change of the flux through a surface such as this one. (The soap film tends to assume a certain special shape which results in the minimum possible surface area, but Faraday's law would be true for any surface that filled in the loop.) In the coiled part of the wire, the soap makes a three-dimensional screw shape, like the shape you would get if you took the steps of a spiral staircase and smoothed them into a ramp. The loop rule is going to be strongly violated for this path.

We can interpret this as follows. Since the wire in the solenoid has a very low resistance compared to the resistances of the light bulbs, we can expect that the electric field along the corkscrew part of loop 1/3 will be very small. As an electron passes through the coil, the work done on it is therefore essentially zero, and the true emf along the coil is zero. In figure 1/1, the meter on top is therefore not telling us the actual emf experienced by an electron that passes through the coil. It is telling us the emf experienced by an electron that passes through the meter itself, which is a different quantity entirely. The other three meters, however, really do tell us the emf through the bulbs, since there are no magnetic fields where they are, and therefore no funny induction effects.
11.6 Maxwell’s equations

11.6.1 Induced magnetic fields

We are almost, but not quite, done figuring out the complete set of physical laws, called Maxwell’s equations, governing electricity and magnetism. We are only missing one more term. For clarity, I’ll state Maxwell’s equations with the missing part included, and then discuss the physical motivation and experimental evidence for sticking it in:

Maxwell’s equations

For any closed surface, the fluxes through the surface are

$$\Phi_E = 4\pi kq_{in}$$

and

$$\Phi_B = 0.$$

For any surface that is not closed, the circulations around the edges of the surface are given by

$$\Gamma_E = -\frac{\partial \Phi_B}{\partial t}$$

and

$$c^2 \Gamma_B = \frac{\partial \Phi_E}{\partial t} + 4\pi k I_{\text{through}}.$$ 

The $\Phi_E$ equation is Gauss’ law: charges make diverging electric fields. The corresponding equation for $\Phi_B$ tells us that magnetic “charges” (monopoles) don’t exist, so magnetic fields never diverge. The third equation says that changing magnetic fields induce curly electric fields, whose curliness we can measure using the emf, $\Gamma_E$, around a closed loop. The final equation, for $\Gamma_B$, is the only one where anything new has been added. Without the new time derivative term, this equation would simply be Ampère’s law. (I’ve chosen to move the $c^2$ over to the left because it simplifies the writing, and also because it more clearly demonstrates the analogous roles played by charges and currents in the $\Phi_E$ and $\Gamma_B$ equations.)

This new $\partial \Phi_E / \partial t$ term says that just as a changing magnetic field can induce a curly electric field, a changing electric field can induce a curly magnetic field. Why should this be so? The following examples show that Maxwell’s equations would not make sense in general without it.

Figure b shows a mysterious curly magnetic field. Magnetic fields are supposed to be made by moving charges, but there don’t seem to be any moving charges in this landscape. Where are they? One reasonable guess would be that they’re behind your head, where you can’t see them. Suppose there’s a positively charged particle about to hit you in the back of the head. This particle is like a current...
going into the page. We’re used to dealing with currents made by many charged particles, but logically we can’t have some minimum number that would qualify as a current. This is not a static current, however, because the current at a given point in space is not staying the same over time. If the particle is pointlike, then it takes zero time to pass any particular location, and the current is then infinite at that point in space. A moment later, when the particle is passing by some other location, there will be an infinite current there, and zero current in the previous location. If this single particle qualifies as a current, then it should be surrounded by a curly magnetic field, just like any other current.\textsuperscript{11}

This explanation is simple and reasonable, but how do we know it’s correct? Well, it makes another prediction, which is that the positively charged particle should be making an electric field as well. Not only that, but if it’s headed for the back of your head, then it’s getting closer and closer, so the electric field should be getting stronger over time. But this is exactly what Maxwell’s equations require. There is no current \( I_{\text{through}} \) piercing the Ampèrian surface shown in figure c, so Maxwell’s equation for \( \Gamma_B \) becomes \( c^2 \Gamma_B = \partial \Phi_E / \partial t \). The only reason for an electric field to change is if there are charged particles making it, and those charged particles are moving. When charged particles are moving, they make magnetic fields as well.

Note that the above example is also sufficient to prove the positive sign of the \( \partial \Phi_E / \partial t \) term in Maxwell’s equations, which is different from the negative sign of Faraday’s \( - \partial \Phi_B / \partial t \) term.

The addition of the \( \partial \Phi_E / \partial t \) term has an even deeper and more important physical meaning. With the inclusion of this term, Maxwell’s equations can describe correctly the way in which disturbances in the electric and magnetic fields ripple outwards at the speed of light. Indeed, Maxwell was the first human to understand that light was in fact an electromagnetic wave. Legend has it that it was on a starry night that he first realized this implication of his equations. He went for a walk with his wife, and told her she was the only other person in the world who really knew what starlight was.

To see how the \( \partial \Phi_E / \partial t \) term relates to electromagnetic waves, let’s look at an example where we would get nonsense without it. Figure d shows an electron that sits just on one side of an imaginary Ampèrian surface, and then hops through it at some randomly

\textsuperscript{11}One way to prove this rigorously is that in a frame of reference where the particle is at rest, it has an electric field that surrounds it on all sides. If the particle has been moving with constant velocity for a long time, then this is just an ordinary Coulomb’s-law field, extending off to very large distances, since disturbances in the field ripple outward at the speed of light. In a frame where the particle is moving, this pure electric field is experienced instead as a combination of an electric field and a magnetic field, so the magnetic field must exist throughout the same vast region of space.
A magnetic field in the form of a sine wave.

A wave pattern is curly. For example, the circulation around this rectangle is nonzero and counterclockwise.

Chosen moment. Unadorned with the $\frac{\partial \Phi_E}{\partial t}$ term, Maxwell’s equation for $\Gamma_B$ reads as $c^2 \Gamma_B = 4\pi k I_{\text{through}}$, which is Ampère’s law. If the electron is a pointlike particle, then we have an infinite current $I_{\text{through}}$ at the moment when it pierces the imaginary surface, and zero current at all other times. An infinite magnetic circulation $\Gamma_B$ can only be produced by an infinite magnetic field, so without the $\frac{\partial \Phi_E}{\partial t}$ term, Maxwell’s equations predict nonsense: the edge of the surface would experience an infinite magnetic field at one instant, and zero magnetic field at all other times. Even if the infinity didn’t upset us, it doesn’t make sense that anything special would happen at the moment the electron passed through the surface, because the surface is an imaginary mathematical construct. We could just as well have chosen the curved surface shown in figure e, which the electron never crosses at all. We are already clearly getting nonsensical results by omitting the $\frac{\partial \Phi_E}{\partial t}$ term, and this shouldn’t surprise us because Ampère’s law only applies to statics. More to the point, Ampère’s law doesn’t have time in it, so it predicts that this effect is instantaneous. According to Ampère’s law, we could send Morse code signals by wiggling the electron back and forth, and these signals would be received at distant locations instantly, without any time delay at all. This contradicts the theory of relativity, one of whose predictions is that information cannot be transmitted at speeds greater than the speed of light.

**Discussion Questions**

A Induced magnetic fields were introduced in the text via the imaginary landscape shown in figure b on page 724, and I argued that the magnetic field could have been produced by a positive charge coming from behind your head. This is a specific assumption about the number of charges (one), the direction of motion, and the sign of the charge. What are some other scenarios that could explain this field?

11.6.2 Light waves

We could indeed send signals using this scheme, and the signals would be a form of light. A radio transmitting antenna, for instance, is simply a device for whipping electrons back and forth at megahertz frequencies. Radio waves are just like visible light, but with a lower frequency. With the addition of the $\frac{\partial \Phi_E}{\partial t}$ term, Maxwell’s equations are capable of describing electromagnetic waves. It would be possible to use Maxwell’s equations to calculate the pattern of the electric and magnetic fields rippling outward from a single electron that fidgets at irregular intervals, but let’s pick a simpler example to analyze.

The simplest wave pattern is a sine wave like the one shown in figure f. Let’s assume a magnetic field of this form, and see what Maxwell’s equations tell us about it. If the wave is traveling through empty space, then there are no charges or currents present,
and Maxwell’s equations become

\[ \Phi_E = 0 \]
\[ \Phi_B = 0 \]
\[ \Gamma_E = -\frac{\partial \Phi_B}{\partial t} \]
\[ c^2 \Gamma_B = \frac{\partial \Phi_E}{\partial t}. \]

The equation \( \Phi = 0 \) has already been verified for this type of wave pattern in example 39 on page 654. Even if you haven’t learned the techniques from that section, it should be visually plausible that this field pattern doesn’t diverge or converge on any particular point.

**Geometry of the electric and magnetic fields**

The equation \( c^2 \Gamma_B = \frac{\partial \Phi_E}{\partial t} \) tells us that there can be no such thing as a purely magnetic wave. The wave pattern clearly does have a nonvanishing circulation around the edge of the surface suggested in figure g, so there must be an electric flux through the surface. This magnetic field pattern must be intertwined with an electric field pattern that fills the same space. There is also no way that the two sides of the equation could stay synchronized with each other unless the electric field pattern is also a sine wave, and one that has the same wavelength, frequency, and velocity. Since the electric field is making a flux through the indicated surface, it’s plausible that the electric field vectors lie in a plane perpendicular to that of the magnetic field vectors. The resulting geometry is shown in figure h. Further justification for this geometry is given later in this subsection.

\[ \text{i / An impossible wave pattern.} \]

\[ \text{h / The geometry of an electromagnetic wave.} \]

One feature of figure h that is easily justified is that the electric and magnetic fields are perpendicular not only to each other, but also to the direction of propagation of the wave. In other words, the vibration is sideways, like people in a stadium “doing the wave,” not lengthwise, like the accordion pattern in figure i. (In standard
wave terminology, we say that the wave is transverse, not longitudinal.) The wave pattern in figure i is impossible, because it diverges from the middle. For virtually any choice of Gaussian surface, the magnetic and electric fluxes would be nonzero, contradicting the equations $\Phi_B = 0$ and $\Phi_E = 0$.\(^\text{12}\)

Reflection

The wave in figure j hits a silvered mirror. The metal is a good conductor, so it has constant voltage throughout, and the electric field equals zero inside it: the wave doesn’t penetrate and is 100% reflected. If the electric field is to be zero at the surface as well, the reflected wave must have its electric field inverted (p. 376), so that the incident and reflected fields cancel there.

But the magnetic field of the reflected wave is not inverted. This is because the reflected wave has to have the correct right-handed relationship between the fields and the direction of propagation.

Polarization

Two electromagnetic waves traveling in the same direction through space can differ by having their electric and magnetic fields in different directions, a property of the wave called its polarization.

The speed of light

What is the velocity of the waves described by Maxwell’s equations? Maxwell convinced himself that light was an electromagnetic wave partly because his equations predicted waves moving at the velocity of light, $c$. The only velocity that appears in the equations is $c$, so this is fairly plausible, although a real calculation is required in order to prove that the velocity of the waves isn’t something like $2c$ or $c/\pi$ — or zero, which is also $c$ multiplied by a constant! The following discussion, leading up to a proof that electromagnetic waves travel at $c$, is meant to be understandable even if you’re reading this book out of order, and haven’t yet learned much about waves. As always with proofs in this book, the reason to read it isn’t to convince yourself that it’s true, but rather to build your intuition. The style will be visual. In all the following figures, the wave patterns are moving across the page (let’s say to the right), and it usually doesn’t matter whether you imagine them as representing the wave’s magnetic field or its electric field, because Maxwell’s equations in a vacuum have the same form for both fields. Whichever field we imagine the figures as representing, the other field is coming in and out of the page.

The velocity of the waves is not zero. If the wave pattern was

\[^{12}\text{Even if the fields can’t be parallel to the direction of propagation, one might wonder whether they could form some angle other than 90 degrees with it. No. One proof is given on page 732. A alternative argument, which is simpler but more esoteric, is that if there was such a pattern, then there would be some other frame of reference in which it would look like figure i.}\]
The velocity of the waves is a fixed number for a given wave pattern. Consider a typical sinusoidal wave of visible light, with a distance of half a micrometer from one peak to the next peak. Suppose this wave pattern provides a valid solution to Maxwell’s equations when it is moving with a certain velocity. We then know, for instance, that there cannot be a valid solution to Maxwell’s equations in which the same wave pattern moves with double that velocity. The time derivatives on the right sides of Maxwell’s equations for $\Gamma_E$ and $\Gamma_B$ would be twice as big, since an observer at a certain point in space would see the wave pattern sweeping past at twice the rate. But the left sides would be the same, so the equations wouldn’t equate.

The velocity is the same for all wave patterns. In other words, it isn’t $0.878c$ for one wave pattern, and $1.067c$ for some other pattern. This is surprising, since, for example, water waves with different shapes do travel at different speeds. Similarly, even though we speak of “the speed of sound,” sound waves do travel at slightly different speeds depending on their pitch and loudness, although the differences are small unless you’re talking about cannon blasts or extremely high frequency ultrasound. To see how Maxwell’s equations give a consistent velocity, consider figure k. Along the right and left edges of the same Ampériean surface, the more compressed wave pattern of blue light has twice as strong a field, so the circulations on the left sides of Maxwell’s equations are twice as large. To satisfy Maxwell’s equations, the time derivatives of the fields must also be twice as large for the blue light. But this is true only if the blue light’s wave pattern is moving to the right at the same speed as the red light’s: if the blue light pattern is sweeping over an observer with a given velocity, then the time between peaks is half as much, like the clicking of the wheels on a train whose cars are half the length.

We can also check that bright and dim light, as shown in figure l, have the same velocity. If you haven’t yet learned much about

\footnote{A young Einstein worried about what would happen if you rode a motorcycle alongside a light wave, traveling at the speed of light. Would the light wave have a zero velocity in this frame of reference? The only solution lies in the theory of relativity, one of whose consequences is that a material object like a student or a motorcycle cannot move at the speed of light.}

\footnote{Actually, this is only exactly true of the rectangular strip is made infinitesimally thin.}

\footnote{You may know already that different colors of light have different speeds when they pass through a material substance, such as the glass or water. This is not in contradiction with what I’m saying here, since this whole analysis is for light in a vacuum.}
waves, then this might be surprising. A material object with more energy goes faster, but that’s not the case for waves. The circulation around the edge of the Ampérian surface shown in the figure is twice as strong for the light whose fields are doubled in strength, so the left sides of Maxwell’s \( \Gamma \) equations are doubled. The right sides are also doubled, because the derivative of twice a function is twice the derivative of the original function. Thus if dim light moving with a particular velocity is a solution, then so is bright light, provided that it has the same velocity.

We can now see that all sinusoidal waves have the same velocity. What about nonsinusoidal waves like the one in figure m? There is a mathematical theorem, due to Fourier, that says any function can be made by adding together sinusoidal functions. For instance, \( 3 \sin x - 7 \cos 3x \) can be made by adding together the functions \( 3 \sin x \) and \( -7 \cos 3x \), but Fourier proved that this can be done even for functions, like figure m, that aren’t obviously built out of sines and cosines in the first place. Therefore our proof that sinusoidal waves all have the same velocity is sufficient to demonstrate that other waves also have this same velocity.

We’re now ready to prove that this universal speed for all electromagnetic waves is indeed \( c \). Since we’ve already convinced ourselves that all such waves travel at the same speed, it’s sufficient to find the velocity of one wave in particular. Let’s pick the wave whose fields have magnitudes

\[
E = \tilde{E} \sin(x + vt) \quad \text{and} \quad B = \tilde{B} \sin(x + vt),
\]

which is about as simple as we can get. The peak electric field of this wave has a strength \( \tilde{E} \), and the peak magnetic field is \( \tilde{B} \). The sine functions go through one complete cycle as \( x \) increases by \( 2\pi = 6.28 \ldots \), so the distance from one peak of this wave to the next — its wavelength — is 6.28… meters. This means that it is not a wave of visible light but rather a radio wave (its wavelength is on the same order of magnitude as the size of a radio antenna). That’s OK. What was glorious about Maxwell’s work was that it unified the whole electromagnetic spectrum. Light is simple. Radio waves aren’t fundamentally any different than light waves, x-rays, or gamma rays.  

The justification for putting \( x + vt \) inside the sine functions is as follows. As the wave travels through space, the whole pattern just shifts over. The fields are zero at \( x = 0, t = 0 \), since the sine of zero is zero. This zero-point of the wave pattern shifts over as time goes by; at any time \( t \) its location is given by \( x + vt = 0 \). After one

\[\footnote{What makes them appear to be unrelated phenomena is that we experience them through their interaction with atoms, and atoms are complicated, so they respond to various kinds of electromagnetic waves in complicated ways.} \]

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second, the zero-point is located at $x = -(1 \text{ s})v$. The distance it travels in one second is therefore numerically equal to $v$, and this is exactly the concept of velocity: how far something goes per unit time.

The wave has to satisfy Maxwell’s equations for $\Gamma_E$ and $\Gamma_B$ regardless of what Ampèrean surfaces we pick, and by applying them to any surface, we could determine the speed of the wave. The surface shown in figure n turns out to result in an easy calculation: a narrow strip of width $2\ell$ and height $h$, coinciding with the position of the zero-point of the field at $t = 0$.

Now let’s apply the equation $c^2 \Gamma_B = \partial \Phi_E/\partial t$ at $t = 0$. Since the strip is narrow, we can approximate the magnetic field using $\sin x \approx x$, which is valid for small $x$. The magnetic field on the right edge of the strip, at $x = \ell$, is then $\tilde{B}\ell$, so the right edge of the strip contributes $\tilde{B}\ell h$ to the circulation. The left edge contributes the same amount, so the left side of Maxwell’s equation is

$$c^2 \Gamma_B = c^2 \cdot 2\tilde{B}\ell h.$$

The other side of the equation is

$$\frac{\partial \Phi_E}{\partial t} = \frac{\partial}{\partial t}(EA) = 2\ell h \frac{\partial E}{\partial t},$$

where we can dispense with the usual sum because the strip is narrow and there is no variation in the field as we go up and down the strip. The derivative equals $v\tilde{E}\cos(x + vt)$, and evaluating the cosine at $x = 0$, $t = 0$ gives

$$\frac{\partial \Phi_E}{\partial t} = 2v\tilde{E}\ell h$$

Maxwell’s equation for $\Gamma_B$ therefore results in

$$2c^2 \tilde{B}\ell h = 2\tilde{E}\ell hv$$

$$c^2 \tilde{B} = v\tilde{E}.$$  

An application of $\Gamma_E = -\partial \Phi_B/\partial t$ gives a similar result, except that there is no factor of $c^2$

$$\tilde{E} = v\tilde{B}.$$  

(The minus sign simply represents the right-handed relationship of the fields relative to their direction of propagation.)
Multiplying these last two equations by each other, we get
\[ c^2 \tilde{B} \tilde{E} = v^2 \tilde{E} \tilde{B} \]
\[ c^2 = v^2 \]
\[ v = \pm c. \]

This is the desired result. (The plus or minus sign shows that the wave can travel in either direction.)

As a byproduct of this calculation, we can find the relationship between the strengths of the electric and magnetic fields in an electromagnetic wave. If, instead of multiplying the equations \( c^2 \tilde{B} = v \tilde{E} \) and \( \tilde{E} = v \tilde{B} \), we divide them, we can easily show that \( \tilde{E} = c \tilde{B} \).

Figure 0 shows the complete spectrum of light waves. The wavelength \( \lambda \) (number of meters per cycle) and frequency \( f \) (number of cycles per second) are related by the equation \( c = f \lambda \). Maxwell’s equations predict that all light waves have the same structure, regardless of wavelength and frequency, so even though radio and x-rays, for example, hadn’t been discovered, Maxwell predicted that such waves would have to exist. Maxwell’s 1865 prediction passed an important test in 1888, when Heinrich Hertz published the results of experiments in which he showed that radio waves could be manipulated in the same ways as visible light waves. Hertz showed, for example, that radio waves could be reflected from a flat surface, and that the directions of the reflected and incoming waves were related in the same way as with light waves, forming equal angles with the surface. Likewise, light waves can be focused with a curved, dish-shaped mirror, and Hertz demonstrated the same thing with radio waves using a metal dish.

**Momentum of light waves**

A light wave consists of electric and magnetic fields, and fields contain energy. Thus a light wave carries energy with it when it travels from one place to another. If a material object has kinetic energy and moves from one place to another, it must also have momentum, so it is logical to ask whether light waves have momentum as well. It can be proved based on relativity\(^\text{17}\) that it does, and that the

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\(^\text{17}\)See problem 11 on p. 460, or example 24 on p. 438.
momentum and energy are related by the equation \( U = \frac{p}{c} \), where \( p \) is the magnitude of the momentum vector, and \( U = U_e + U_m \) is the sum of the energy of the electric and magnetic fields. We can now demonstrate this without explicitly referring to relativity, and connect it to the specific structure of a light wave.

The energy density of a light wave is related to the magnitudes of the fields in a specific way — it depends on the squares of their magnitudes, \( E^2 \) and \( B^2 \), which are the same as the dot products \( \mathbf{E} \cdot \mathbf{E} \) and \( \mathbf{B} \cdot \mathbf{B} \). We argued on page 606 that since energy is a scalar, the only possible expressions for the energy densities of the fields are dot products like these, multiplied by some constants. This is because the dot product is the only mathematically sensible way of multiplying two vectors to get a scalar result. (Any other way violates the symmetry of space itself.)

How does this relate to momentum? Well, we know that if we double the strengths of the fields in a light beam, it will have four times the energy, because the energy depends on the square of the fields. But we then know that this quadruple-energy light beam must have quadruple the momentum as well. If there wasn’t this kind of consistency between the momentum and the energy, then we could violate conservation of momentum by combining light beams or splitting them up. We therefore know that the momentum density of a light beam must depend on a field multiplied by a field. Momentum, however, is a vector, and there is only one physically meaningful way of multiplying two vectors to get a vector result, which is the cross product (see page 1027). The momentum density can therefore only depend on the cross products \( \mathbf{E} \times \mathbf{E} \), \( \mathbf{B} \times \mathbf{B} \), and \( \mathbf{E} \times \mathbf{B} \). But the first two of these are zero, since the cross product vanishes when there is a zero angle between the vectors. Thus the momentum per unit volume must equal \( \mathbf{E} \times \mathbf{B} \) multiplied by some constant,

\[
dp = (\text{constant}) \mathbf{E} \times \mathbf{B} \ dv
\]

This predicts something specific about the direction of propagation of a light wave: it must be along the line perpendicular to the electric and magnetic fields. We’ve already seen that this is correct, and also that the electric and magnetic fields are perpendicular to each other. Therefore this cross product has a magnitude

\[
|\mathbf{E} \times \mathbf{B}| = |\mathbf{E}| |\mathbf{B}| \sin 90^\circ
= |\mathbf{E}| |\mathbf{B}|
= \frac{|\mathbf{E}|^2}{c} = c|\mathbf{B}|^2,
\]

where in the last step the relation \( |\mathbf{E}| = c|\mathbf{B}| \) has been used.

We now only need to find one physical example in order to fix the constant of proportionality. Indeed, if we didn’t know relativity, it would be possible to believe that the constant of proportionality was
p / A classical calculation of the momentum of a light wave. An antenna of length \( \ell \) is bathed in an electromagnetic wave. The black arrows represent the electric field, the white circles the magnetic field coming out of the page. The wave is traveling to the right.

q / A simplified drawing of the 1903 experiment by Nichols and Hull that verified the predicted momentum of light waves. Two circular mirrors were hung from a fine quartz fiber, inside an evacuated bell jar. A 150 mW beam of light was shone on one of the mirrors for 6 s, producing a tiny rotation, which was measurable by an optical lever (not shown). The force was within 0.6% of the theoretically predicted value (problem 11 on p. 460) of 0.001 \( \mu \)N. For comparison, a short clipping of a single human hair weighs \( \sim \) 1 \( \mu \)N.

---

The simplest example of which I know is as follows. Suppose a piece of wire of length \( \ell \) is bathed in electromagnetic waves coming in sideways, and let’s say for convenience that this is a radio wave, with a wavelength that is large compared to \( \ell \), so that the fields don’t change significantly across the length of the wire. Let’s say the electric field of the wave happens to be aligned with the wire. Then there is an emf between the ends of the wire which equals \( E\ell \), and since the wire is small compared to the wavelength, we can pretend that the field is uniform, not curly, in which case voltage is a well-defined concept, and this is equivalent to a voltage difference \( \Delta V = E\ell \) between the ends of the wire. The wire obeys Ohm’s law, and a current flows in response to the wave.\(^{18}\) Equating the expressions \( dU/dt \) and \( I\Delta V \) for the power dissipated by ohmic heating, we have

\[
dU = IE\ell \, dt
\]

for the energy the wave transfers to the wire in a time interval \( dt \).

Note that although some electrons have been set in motion in the wire, we haven’t yet seen any momentum transfer, since the protons are experiencing the same amount of electric force in the opposite direction. However, the electromagnetic wave also has a magnetic field, and a magnetic field transfers momentum to (exerts a force on) a current. This is only a force on the electrons, because they’re what make the current. The magnitude of this force equals \( \ell IB \) (homework problem 6), and using the definition of force, \( dp/dt \), we find for the magnitude of the momentum transferred:

\[
dp = \ell IB \, dt
\]

We now know both the amount of energy and the amount of momentum that the wave has lost by interacting with the wire. Dividing these two equations, we find

\[
\frac{dp}{dU} = \frac{B}{E} = \frac{1}{c},
\]

which is what we expected based on relativity. This can now be restated in the form \( dp = (\text{constant})E \times B \, dv \) (homework problem 40).

Note that although the equations \( p = U/c \) and \( dp = (\text{constant})E \times B \, dv \) are consistent with each other for a sine wave, they are not consistent with each other in general. The relativistic argument leading up to \( p = U/c \) assumed that we were only talking about

---

\(^{18}\)This current will soon come to a grinding halt, because we don’t have a complete circuit, but let’s say we’re talking about the first picosecond during which the radio wave encounters the wire.
a single thing traveling in a single direction, whereas no such assumption was made in arguing for the $\mathbf{E} \times \mathbf{B}$ form. For instance, if two light beams of equal strength are traveling through one another, going in opposite directions, their total momentum is zero, which is consistent with the $\mathbf{E} \times \mathbf{B}$ form, but not with $U/c$.

Some examples were given in chapter 3 of situations where it actually matters that light has momentum. Figure q shows the first confirmation of this fact in the laboratory.

**Angular momentum of light waves**

For completeness, we note that since light carries momentum, it must also be possible for it to have angular momentum. If you’ve studied chemistry, here’s an example of why this can be important. You know that electrons in atoms can exist in states labeled s, p, d, f, and so on. What you might not have realized is that these are angular momentum labels. The s state, for example, has zero angular momentum. If light didn’t have angular momentum, then, for example, it wouldn’t be possible for a hydrogen atom in a p state to change to the lower-energy s state by emitting light. Conservation of angular momentum requires that the light wave carry away all the angular momentum originally possessed by the electron in the p state, since in the s state it has none.

**Discussion Questions**

**A** Positive charges 1 and 2 are moving as shown. What electric and magnetic forces do they exert on each other? What does this imply for conservation of momentum?

**B** 1. The figure shows a line of charges moving to the right, creating a current $I$. An Ampèrian surface in the form of a disk has been superimposed. Use Maxwell’s equations to find the field $\mathbf{B}$ at point P.

2. A tiny gap is chopped out of the line of charge. What happens when this gap is directly underneath the point P?

**C** The diagram shows an electric field pattern frozen at one moment in time. Let’s imagine that it’s the electric part of an electromagnetic wave. Consider four possible directions in which it could be propagating: left, right, up, and down. Determine whether each of these is consistent with Maxwell’s equations. If so, infer the direction of the magnetic field.

**D** What happens if we use Maxwell’s equations to analyze the behavior of the wave in a frame of reference moving along with the wave?
11.7 Electromagnetic properties of materials

Different types of matter have a variety of useful electrical and magnetic properties. Some are conductors, and some are insulators. Some, like iron and nickel, can be magnetized, while others have useful electrical properties, e.g., dielectrics, discussed qualitatively in the discussion question on page 616, which allow us to make capacitors with much higher values of capacitance than would otherwise be possible. We need to organize our knowledge about the properties that materials can possess, and see whether this knowledge allows us to calculate anything useful with Maxwell’s equations.

11.7.1 Conductors

A perfect conductor, such as a superconductor, has no DC electrical resistance. It is not possible to have a static electric field inside it, because then charges would move in response to that field, and the motion of the charges would tend to reduce the field, contrary to the assumption that the field was static. Things are a little different at the surface of a perfect conductor than on the interior. We expect that any net charges that exist on the conductor will spread out under the influence of their mutual repulsion, and settle on the surface. As we saw in chapter 10, Gauss’s law requires that the fields on the two sides of a sheet of charge have $|E_{\perp,1} - E_{\perp,2}|$ proportional to the surface charge density, and since the field inside the conductor is zero, we infer that there can be a field on or immediately outside the conductor, with a nonvanishing component perpendicular to the surface. The component of the field parallel to the surface must vanish, however, since otherwise it would cause the charges to move along the surface.

On a hot summer day, the reason the sun feels warm on your skin is that the oscillating fields of the light waves excite currents in your skin, and these currents dissipate energy by ohmic heating. In a perfect conductor, however, this could never happen, because there is no such thing as ohmic heating. Since electric fields can’t penetrate a perfect conductor, we also know that an electromagnetic wave can never pass into one. By conservation of energy, we know that the wave can’t just vanish, and if the energy can’t be dissipated as heat, then the only remaining possibility is that all of the wave’s energy is reflected. This is why metals, which are good electrical conductors, are also highly reflective. They are not perfect electrical conductors, however, so they are not perfectly reflective. The wave enters the conductor, but immediately excites oscillating currents, and these oscillating currents dissipate the energy both by ohmic heating and by reradiating the reflected wave. Since the parts of Maxwell’s equations describing radiation have time derivatives in them, the efficiency of this reradiation process depends strongly on frequency. When the frequency is high and the material is a good conductor, reflection predominates, and is so efficient that the wave
only penetrates to a very small depth, called the skin depth. In the limit of poor conduction and low frequencies, absorption predominates, and the skin depth becomes much greater. In a high-frequency AC circuit, the skin depth in a copper wire is very small, and therefore the signals in such a circuit are propagated entirely at the surfaces of the wires. In the limit of low frequencies, i.e., DC, the skin depth approaches infinity, so currents are carried uniformly over the wires’ cross-sections.

We can quantify how well a particular material conducts electricity. We know that the resistance of a wire is proportional to its length, and inversely proportional to its cross-sectional area. The constant of proportionality is $1/\sigma$, where $\sigma$ (not the same $\sigma$ as the surface charge density) is called the electrical conductivity. Exposed to an electric field $E$, a conductor responds with a current per unit cross-sectional area $J = \sigma E$. The skin depth is proportional to $1/\sqrt{f\sigma}$, where $f$ is the frequency of the wave.

### 11.7.2 Dielectrics

A material with a very low conductivity is an insulator. Such materials are usually composed of atoms or molecules whose electrons are strongly bound to them; since the atoms or molecules have zero total charge, their motion cannot create an electric current. But even though they have zero charge, they may not have zero dipole moment. Imagine such a substance filling in the space between the plates of a capacitor, as in figure a. For simplicity, we assume that the molecules are oriented randomly at first, a/1, and then become completely aligned when a field is applied, a/2. The effect has been to take all of the negatively charged black ends of the molecules and shift them upward, and the opposite for the positively charged white ends. Where the black and white charges overlap, there is still zero net charge, but we have a strip of negative charge at the top, and a strip of positive charge at the bottom, a/3. The effect has been to cancel out part of the charge that was deposited on the plates of the capacitor. Now this is very subtle, because Maxwell’s equations treat these charges on an equal basis, but in terms of practical measurements, they are completely different. The charge on the plates can be measured be inserting an ammeter in the circuit, and integrating the current over time. But the charges in the layers at the top and bottom of the dielectric never flowed through any wires, and cannot be detected by an ammeter. In other words, the total charge, $q$, appearing in Maxwell’s equations is actually $q = q_{\text{free}} - q_{\text{bound}}$, where $q_{\text{free}}$ is the charge that moves freely through wires, and can be detected in an ammeter, while $q_{\text{bound}}$ is the charge bound onto the individual molecules, which can’t. We will, however, detect the presence of the bound charges via their electric fields. Since their electric fields partially cancel the fields of the free charges, a voltmeter will register a smaller than expected
A stud finder is used to locate the wooden beams, or studs, that form the frame behind the wallboard. It is a capacitor whose capacitance changes when it is brought close to a substance with a particular permittivity. Although the wall is external to the capacitor, a change in capacitance is still observed, because the capacitor has “fringing fields” that extend outside the region between its plates.

Although the relationship $E \leftrightarrow q$ between electric fields and their sources is unalterably locked in by Gauss’s law, that’s not what we see in practical measurements. In this example, we can measure the voltage difference between the plates of the capacitor and divide by the distance between them to find $E$, and then integrate an ammeter reading to find $q_{\text{free}}$, and we will find that Gauss’s law appears not to hold. We have $E \leftrightarrow q_{\text{free}}/(\text{constant})$, where the constant fudge factor is greater than one. This constant is a property of the dielectric material, and tells us how many dipoles there are, how strong they are, and how easily they can be reoriented. The conventional notation is to incorporate this fudge factor into Gauss’s law by defining an altered version of the electric field,

$$D = \epsilon E,$$

and to rewrite Gauss’s law as

$$\Phi_D = q_{\text{in, free}}.$$ 

The constant $\epsilon$ is a property of the material, known as its permittivity. In a vacuum, $\epsilon$ takes on a value known as $\epsilon_o$, defined as $1/(4\pi k)$. In a dielectric, $\epsilon$ is greater than $\epsilon_o$. When a dielectric is present between the plates of a capacitor, its capacitance is proportional to $\epsilon$ (problem 38). The following table gives some sample values of the permittivities of a few substances.

<table>
<thead>
<tr>
<th>substance</th>
<th>$\epsilon/\epsilon_o$ at zero frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>vacuum</td>
<td>1</td>
</tr>
<tr>
<td>air</td>
<td>1.00054</td>
</tr>
<tr>
<td>water</td>
<td>80</td>
</tr>
<tr>
<td>barium titanate</td>
<td>1250</td>
</tr>
</tbody>
</table>

A capacitor with a very high capacitance is potentially a superior replacement for a battery, but until the 1990’s this was impractical because capacitors with high enough values couldn’t be made, even with dielectrics having the largest known permittivities. Such supercapacitors, some with values in the kilofarad range, are now available. Most of them do not use dielectric at all; the very high capacitance values are instead obtained by using electrodes that are not parallel metal plates at all, but exotic materials such as aerogels, which allows the spacing between the “electrodes” to be very small.

Although figure a/2 shows the dipoles in the dielectric being completely aligned, this is not a situation commonly encountered in practice. In such a situation, the material would be as polarized as it could possibly be, and if the field was increased further, it would not respond. In reality, a capacitor, for example, would normally be operated with fields that produced quite a small amount of alignment, and it would be under these conditions that the linear
relationship \( \mathbf{D} = \varepsilon \mathbf{E} \) would actually be a good approximation. Before a material’s maximum polarization is reached, it may actually spark or burn up.

**self-check I**

Suppose a parallel-plate capacitor is built so that a slab of dielectric material can be slid in or out. (This is similar to the way the stud finder in figure b works.) We insert the dielectric, hook the capacitor up to a battery to charge it, and then use an ammeter and a voltmeter to observe what happens when the dielectric is withdrawn. Predict the changes observed on the meters, and correlate them with the expected change in capacitance. Discuss the energy transformations involved, and determine whether positive or negative work is done in removing the dielectric.

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11.7.3 **Magnetic materials**

**Magnetic permeability**

Atoms and molecules may have magnetic dipole moments as well as electric dipole moments. Just as an electric dipole contains bound charges, a magnetic dipole has bound currents, which come from the motion of the electrons as they orbit the nucleus, \( c/1 \). Such a substance, subjected to a magnetic field, tends to align itself, \( c/2 \), so that a sheet of current circulates around the externally applied field. Figure \( c/3 \) is closely analogous to figure \( a/3 \); in the central gray area, the atomic currents cancel out, but the atoms at the outer surface form a sheet of bound current. However, whereas like charges repel and opposite charges attract, it works the other way around for currents: currents in the same direction attract, and currents in opposite directions repel. Therefore the bound currents in a material inserted inside a solenoid tend to reinforce the free currents, and the result is to strengthen the field. The total current is \( I = I_{\text{free}} + I_{\text{bound}} \), and we define an altered version of the magnetic field,

\[
\mathbf{H} = \frac{\mathbf{B}}{\mu},
\]

and rewrite Ampère’s law as

\[
\Gamma_H = I_{\text{through, free}}.
\]

The constant \( \mu \) is the permeability, with a vacuum value of \( \mu_0 = 4\pi k/c^2 \). Here are the magnetic permeabilities of some substances:

<table>
<thead>
<tr>
<th>substance</th>
<th>( \mu/\mu_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>vacuum</td>
<td>1</td>
</tr>
<tr>
<td>aluminum</td>
<td>1.00002</td>
</tr>
<tr>
<td>steel</td>
<td>700</td>
</tr>
<tr>
<td>transformer iron</td>
<td>4,000</td>
</tr>
<tr>
<td>mu-metal</td>
<td>20,000</td>
</tr>
</tbody>
</table>

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**d** / Example 24: a cutaway view of a solenoid.
Example 24: without the iron core, the field is so weak that it barely deflects the compass. With it, the deflection is nearly 90°.

A transformer with a laminated iron core. The input and output coils are inside the paper wrapper. The iron core is the black part that passes through the coils at the center, and also wraps around them on the outside.

Example 25: ferrite beads. The top panel shows a clip-on type, while the bottom shows one built into a cable.

An iron-core electromagnet example 24

A solenoid has 1000 turns of wire wound along a cylindrical core with a length of 10 cm. If a current of 1.0 A is used, find the magnetic field inside the solenoid if the core is air, and if the core is made of iron with $\mu/\mu_0 = 4,000$.

Air has essentially the same permability as vacuum, so using the result of example 13 on page 703, we find that the field is 0.013 T.

We now consider the case where the core is filled with iron. The original derivation in example 13 started from Ampère’s law, which we now rewrite as $\Gamma_H = I_{\text{through, free}}$. As argued previously, the only significant contributions to the circulation come from line segment AB. This segment lies inside the iron, where $H = B/\mu$. The field is the same as in the air-core case, since the new form of Ampère’s law only relates $H$ to the current in the wires (the free current). This means that $B = \mu H$ is greater by a factor of 4,000 than in the air-core case, or 52 T. This is an extremely intense field — so intense, in fact, that the iron’s magnetic polarization would probably become saturated before we could actually get the field that high.

The electromagnet of example 24 could also be used as an inductor, and its inductance would be proportional to the permittivity of the core. This makes it possible to construct high-value inductors that are relatively compact. Permeable cores are also used in transformers.

A transformer or inductor with a permeable core does have some disadvantages, however, in certain applications. The oscillating magnetic field induces an electric field, and because the core is typically a metal, these currents dissipate energy strongly as heat. This behaves like a fairly large resistance in series with the coil. Figure f shows a method for reducing this effect. The iron core of this transformer has been constructed out of laminated layers, which has the effect of blocking the conduction of the eddy currents.

A ferrite bead example 25

Cables designed to carry audio signals are typically made with two adjacent conductors, such that the current flowing out through one conductor comes back through the other one. Computer cables are similar, but usually have several such pairs bundled inside the insulator. This paired arrangement is known as differential mode, and has the advantage of cutting down on the reception and transmission of interference. In terms of transmission, the magnetic field created by the outgoing current is almost exactly canceled by the field from the return current, so electromagnetic waves are only weakly induced. In reception, both conductors are bathed in the same electric and magnetic fields, so an emf that adds current on one side subtracts current from the other side,
resulting in cancellation.

The opposite of differential mode is called common mode. In common mode, all conductors have currents flowing in the same direction. Even when a circuit is designed to operate in differential mode, it may not have exactly equal currents in the two conductors with $I_1 + I_2 = 0$, meaning that current is leaking off to ground at one end of the circuit or the other. Although paired cables are relatively immune to differential-mode interference, they do not have any automatic protection from common-mode interference.

Figure g shows a device for reducing common-mode interference called a ferrite bead, which surrounds the cable like a bead on a string. Ferrite is a magnetically permeable alloy. In this application, the ohmic properties of the ferrite actually turn out to be advantageous.

Let’s consider common-mode transmission of interference. The bare cable has some DC resistance, but is also surrounded by a magnetic field, so it has inductance as well. This means that it behaves like a series L-R circuit, with an impedance that varies as $R + i\omega L$, where both $R$ and $L$ are very small. When we add the ferrite bead, the inductance is increased by orders of magnitude, but so is the resistance. Neither $R$ nor $L$ is actually constant with respect to frequency, but both are much greater than for the bare cable.

Suppose, for example, that a signal is being transmitted from a digital camera to a computer via a USB cable. The camera has an internal impedance that is on the order of 10 $\Omega$, the computer’s input also has a $\sim 10$ $\Omega$ impedance, and in differential mode the ferrite bead has no effect, so the cable’s impedance has its low, designed value (probably also about 10 $\Omega$, for good impedance matching). The signal is transmitted unattenuated from the camera to the computer, and there is almost no radiation from the cable.

But in reality there will be a certain amount of common-mode current as well. With respect to common mode, the ferrite bead has a large impedance, with the exact value depending on frequency, but typically on the order of 100 $\Omega$ for frequencies in the MHz range. We now have a series circuit consisting of three impedances: 10, 100, and 10 $\Omega$. For a given emf applied by an external radio wave, the current induced in the circuit has been attenuated by an order of magnitude, relative to its value without the ferrite bead.

Why is the ferrite necessary at all? Why not just insert ordinary air-core inductors in the circuit? We could, for example, have two solenoidal coils, one in the outgoing line and one in the return line, interwound with one another with their windings oriented so that
A frog is levitated diamagnetically by the nonuniform field inside a powerful magnet. Evidently frog has $\mu < \mu_0$.

At a boundary between two substances with $\mu_2 > \mu_1$, the $H$ field has a continuous component parallel to the surface, which implies a discontinuity in the parallel component of the magnetic field $B$.

Their differential-mode fields would cancel. There are two good reasons to prefer the ferrite bead design. One is that it allows a clip-on device like the one in the top panel of figure g, which can be added without breaking the circuit. The other is that our circuit will inevitably have some stray capacitance, and will therefore act like an LRC circuit, with a resonance at some frequency. At frequencies close to the resonant frequency, the circuit would absorb and transmit common-mode interference very strongly, which is exactly the opposite of the effect we were hoping to produce. The resonance peak could be made low and broad by adding resistance in series, but this extra resistance would attenuate the differential-mode signals as well as the common-mode ones. The ferrite's resistance, however, is actually a purely magnetic effect, so it vanishes in differential mode.

Surprisingly, some materials have magnetic permeabilities less than $\mu_0$. This cannot be accounted for in the model above, and although there are semiclassical arguments that can explain it to some extent, it is fundamentally a quantum mechanical effect. Materials with $\mu > \mu_0$ are called paramagnetic, while those with $\mu < \mu_0$ are referred to as diamagnetic. Diamagnetism is generally a much weaker effect than paramagnetism, and is easily masked if there is any trace of contamination from a paramagnetic material. Diamagnetic materials have the interesting property that they are repelled from regions of strong magnetic field, and it is therefore possible to levitate a diamagnetic object above a magnet, as in figure h.

A complete statement of Maxwell's equations in the presence of electric and magnetic materials is as follows:

$$\Phi_D = q_{\text{free}}$$
$$\Phi_B = 0$$
$$\Gamma_E = -\frac{d\Phi_B}{dt}$$
$$\Gamma_H = \frac{d\Phi_D}{dt} + I_{\text{free}}$$

Comparison with the vacuum case shows that the speed of an electromagnetic wave moving through a substance described by permittivity and permeability $\varepsilon$ and $\mu$ is $1/\sqrt{\varepsilon\mu}$. For most substances, $\mu \approx \mu_0$, and $\varepsilon$ is highly frequency-dependent.

Suppose we have a boundary between two substances. By constructing a Gaussian or Ampèrian surface that extends across the boundary, we can arrive at various constraints on how the fields must behave as me move from one substance into the other, when there are no free currents or charges present, and the fields are static. An interesting example is the application of Faraday's law, $\Gamma_H = 0$, to the case where one medium — let's say it's air — has...
a low permeability, while the other one has a very high one. We will violate Faraday’s law unless the component of the $\mathbf{H}$ field parallel to the boundary is a continuous function, $H_{∥,1} = H_{∥,2}$. This means that if $\mu/\mu_0$ is very high, the component of $\mathbf{B} = \mu \mathbf{H}$ parallel to the surface will have an abrupt discontinuity, being much stronger inside the high-permeability material. The result is that when a magnetic field enters a high-permeability material, it tends to twist abruptly to one side, and the pattern of the field tends to be channeled through the material like water through a funnel. In a transformer, a permeable core functions to channel more of the magnetic flux from the input coil to the output coil. Figure j shows another example, in which the effect is to shield the interior of the sphere from the externally imposed field. Special high-permeability alloys, with trade names like Mu-Metal, are sold for this purpose.

![Diagram of a hollow sphere with $\mu/\mu_0 = 10$, immersed in a uniform, externally imposed magnetic field. The interior of the sphere is shielded from the field. The arrows map the magnetic field $\mathbf{B}$. (See homework problem 47, page 757.)](image)
Ferromagnetism

The very last magnetic phenomenon we’ll discuss is probably the very first experience you ever had of magnetism. Ferromagnetism is a phenomenon in which a material tends to organize itself so that it has a nonvanishing magnetic field. It is exhibited strongly by iron and nickel, which explains the origin of the name.

Figure k/1 is a simple one-dimensional model of ferromagnetism. Each magnetic compass needle represents an atom. The compasses in the chain are stable when aligned with one another, because each one’s north end is attracted to its neighbor’s south end. The chain can be turned around, k/2, without disrupting its organization, and the compasses do not realign themselves with the Earth’s field, because their torques on one another are stronger than the Earth’s torques on them. The system has a memory. For example, if I want to remind myself that my friend’s address is 137 Coupling Ct., I can align the chain at an angle of 137 degrees. The model fails, however, as an explanation of real ferromagnetism, because in two or more dimensions, the most stable arrangement of a set of interacting magnetic dipoles is something more like k/3, in which alternating rows point in opposite directions. In this two-dimensional pattern, every compass is aligned in the most stable way with all four of its neighbors. This shows that ferromagnetism, like diamagnetism, has no purely classical explanation; a full explanation requires quantum mechanics.

Because ferromagnetic substances “remember” the history of how they were prepared, they are commonly used to store information in computers. Figure l shows 16 bits from an ancient (ca. 1970) 4-kilobYTE random-access memory, in which each doughnut-shaped iron “core” can be magnetized in one of two possible directions, so that it stores one bit of information. Today, RAM is made of transistors rather than magnetic cores, but a remnant of the old technology remains in the term “core dump,” meaning “memory dump,” as in “my girlfriend gave me a total core dump about her mom’s divorce.” Most computer hard drives today do store their information on rotating magnetic platters, but the platter technology...
may be obsoleted by flash memory in the near future.

The memory property of ferromagnets can be depicted on the type of graph shown in figure m, known as a hysteresis curve. The y axis is the magnetization of a sample of the material — a measure of the extent to which its atomic dipoles are aligned with one another. If the sample is initially unmagnetized, 1, and a field \( H \) is externally applied, the magnetization increases, 2, but eventually becomes saturated, 3, so that higher fields do not result in any further magnetization, 4. The external field can then be reduced, 5, and even eliminated completely, but the material will retain its magnetization. It is a permanent magnet. To eliminate its magnetization completely, a substantial field must be applied in the opposite direction. If this reversed field is made stronger, then the substance will eventually become magnetized just as strongly in the opposite direction. Since the hysteresis curve is nonlinear, and is not a function (it has more than one value of \( M \) for a particular value of \( B \)), a ferromagnetic material does not have a single, well-defined value of the permeability \( \mu \); a value like 4,000 for transformer iron represents some kind of a rough average.

\[ \text{The fluxgate compass} \quad \text{example 26} \]

The fluxgate compass is a type of magnetic compass without moving parts, commonly used on ships and aircraft. An AC current is applied in a coil wound around a ferromagnetic core, driving the core repeatedly around a hysteresis loop. Because the hysteresis curve is highly nonlinear, the addition of an external field such as the Earth’s alters the core’s behavior. Suppose, for example, that the axis of the coil is aligned with the magnetic north-south. The core will reach saturation more quickly when the coil’s field is in the same direction as the Earth’s, but will not saturate as early in the next half-cycle, when the two fields are in opposite directions. With the use of multiple coils, the components of the Earth’s field can be measured along two or three axes, permitting the compass’s orientation to be determined in two or (for aircraft) three dimensions.

\[ \text{Sharp magnet poles} \quad \text{example 27} \]

Although a ferromagnetic material does not really have a single value of the magnetic permeability, there is still a strong tendency to have \( B_\parallel \approx 0 \) just outside the magnet’s surface, for the same reasons as discussed above for high-permeability substances in general. For example, if we have a cylindrical bar magnet about the size and shape of your finger, magnetized lengthwise, then the field near the ends is nearly perpendicular to the surfaces, while the field near the sides, although it may be oriented nearly parallel to the surface, is very weak, so that we still have \( B_\parallel \approx 0 \). This is in close analogy to the situation for the electric field near the surface of a conductor in equilibrium, for which \( E_\parallel = 0 \).
This analogy is close enough so that we can recycle much of our knowledge about electrostatics.

For example, we saw in example 9, p. 546, and problem 37, p. 572, that charge tends to collect on the most highly curved portions of a conductor, and therefore becomes especially dense near a corner or knife-edge. This gives us a way of making especially intense magnetic fields. Most people would imagine that a very intense field could be made simply by using a very large and bulky permanent magnet, but this doesn’t actually work very well, because magnetic dipole fields fall off as $1/r^3$, so that at a point near the surface, nearly all the field is contributed by atoms near the surface. Our analogy with electrostatics suggests that we should instead construct a permanent magnet with a sharp edge.

Figure 0 shows the cross-sectional shapes of two magnet poles used in the historic Stern-Gerlach experiment (sec. 14.1, p. 959). The external magnetic field is represented using field lines. The field lines enter and exit the surfaces perpendicularly, and they are particularly dense near the corner of the upper pole, indicating a strong field. The spreading of the field lines indicates that the field is strongly nonuniform, becoming much weaker toward the bottom of the gap between the poles. This strong nonuniformity was crucial for the experiment, in which the magnets were used as part of a dipole spectrometer. See example 7 and figure p on p. 591 for an explanation of an electric version of such a spectrometer.

This chapter is summarized on page 1089. Notation and terminology are tabulated on pages 1070-1071.
Problems

The symbols √, ⊥, etc. are explained on page 761.

1 A particle with a charge of 1.0 C and a mass of 1.0 kg is observed moving past point P with a velocity (1.0 m/s)\(\hat{x}\). The electric field at point P is (1.0 V/m)\(\hat{y}\), and the magnetic field is (2.0 T)\(\hat{y}\). Find the force experienced by the particle. √

2 For a positively charged particle moving through a magnetic field, the directions of the \(\mathbf{v}\), \(\mathbf{B}\), and \(\mathbf{F}\) vectors are related by a right-hand rule:

\(\mathbf{v}\) along the fingers, with the hand flat
\(\mathbf{B}\) along the fingers, with the knuckles bent
\(\mathbf{F}\) along the thumb

Make a three-dimensional model of the three vectors using pencils or rolled-up pieces of paper to represent the vectors assembled with their tails together. Make all three vectors perpendicular to each other. Now write down every possible way in which the rule could be rewritten by scrambling up the three symbols \(\mathbf{v}\), \(\mathbf{B}\), and \(\mathbf{F}\). Referring to your model, which are correct and which are incorrect? □

3 A charged particle is released from rest. We see it start to move, and as it gets going, we notice that its path starts to curve. Can we tell whether this region of space has \(\mathbf{E} \neq 0\), or \(\mathbf{B} \neq 0\), or both? Assume that no other forces are present besides the possible electrical and magnetic ones, and that the fields, if they are present, are uniform. □

4 A charged particle is in a region of space in which there is a uniform magnetic field \(\mathbf{B} = B\hat{z}\). There is no electric field, and no other forces act on the particle. In each case, describe the future motion of the particle, given its initial velocity.

(a) \(\mathbf{v}_0 = 0\)
(b) \(\mathbf{v}_0 = (1 \text{ m/s})\hat{z}\)
(c) \(\mathbf{v}_0 = (1 \text{ m/s})\hat{y}\)

5 (a) A line charge, with charge per unit length \(\lambda\), moves at velocity \(v\) along its own length. How much charge passes a given point in time \(dt\)? What is the resulting current? △ Answer, p. 1069
(b) Show that the units of your answer in part a work out correctly.

Remark: This constitutes a physical model of an electric current, and it would be a physically realistic model of a beam of particles moving in a vacuum, such as the electron beam in a television tube. It is not a physically realistic model of the motion of the electrons in a current-carrying wire, or of the ions in your nervous system; the motion of the charge carriers in these systems is much more complicated and chaotic, and there are charges of both signs, so that the total charge is zero. But even when the model is physically unrealistic, it still gives the right answers when you use it to compute magnetic effects. This is a remarkable fact, which we will not prove. The interested reader is referred to E.M. Purcell,
6 Two parallel wires of length $L$ carry currents $I_1$ and $I_2$. They are separated by a distance $R$, and we assume $R$ is much less than $L$, so that our results for long, straight wires are accurate. The goal of this problem is to compute the magnetic forces acting between the wires.

(a) Neither wire can make a force on itself. Therefore, our first step in computing wire 1’s force on wire 2 is to find the magnetic field made only by wire 1, in the space occupied by wire 2. Express this field in terms of the given quantities.

(b) Let’s model the current in wire 2 by pretending that there is a line charge inside it, possessing density per unit length $\lambda_2$ and moving at velocity $v_2$. Relate $\lambda_2$ and $v_2$ to the current $I_2$, using the result of problem 5a. Now find the magnetic force wire 1 makes on wire 2, in terms of $I_1$, $I_2$, $L$, and $R$.

Answer, p. 1069

(c) Show that the units of the answer to part b work out to be newtons.

7 Suppose a charged particle is moving through a region of space in which there is an electric field perpendicular to its velocity vector, and also a magnetic field perpendicular to both the particle’s velocity vector and the electric field. Show that there will be one particular velocity at which the particle can be moving that results in a total force of zero on it; this requires that you analyze both the magnitudes and the directions of the forces compared to one another. Relate this velocity to the magnitudes of the electric and magnetic fields. (Such an arrangement, called a velocity filter, is one way of determining the speed of an unknown particle.)

8 The following data give the results of two experiments in which charged particles were released from the same point in space, and the forces on them were measured:

$q_1 = 1 \mu C$, \hspace{1cm} $v_1 = (1 \text{ m/s}) \hat{x}$, \hspace{1cm} $F_1 = (-1 \text{ mN}) \hat{y}$

$q_2 = -2 \mu C$, \hspace{1cm} $v_2 = (-1 \text{ m/s}) \hat{x}$, \hspace{1cm} $F_2 = (-2 \text{ mN}) \hat{y}$

The data are insufficient to determine the magnetic field vector; demonstrate this by giving two different magnetic field vectors, both of which are consistent with the data.

9 The following data give the results of two experiments in which charged particles were released from the same point in space, and the forces on them were measured:

$q_1 = 1 \text{ nC}$, \hspace{1cm} $v_1 = (1 \text{ m/s}) \hat{z}$, \hspace{1cm} $F_1 = (5 \text{ pN}) \hat{x} + (2 \text{ pN}) \hat{y}$

$q_2 = 1 \text{ nC}$, \hspace{1cm} $v_2 = (3 \text{ m/s}) \hat{z}$, \hspace{1cm} $F_2 = (10 \text{ pN}) \hat{x} + (4 \text{ pN}) \hat{y}$

Is there a nonzero electric field at this point? A nonzero magnetic field?

10 This problem is a continuation of problem 6. Note that the answer to problem 6b is given on page 1069.

(a) Interchanging the 1’s and 2’s in the answer to problem 6b, what
is the magnitude of the magnetic force from wire 2 acting on wire 1? Is this consistent with Newton’s third law?

(b) Suppose the currents are in the same direction. Make a sketch, and use the right-hand rule to determine whether wire 1 pulls wire 2 towards it, or pushes it away.

(c) Apply the right-hand rule again to find the direction of wire 2’s force on wire 1. Does this agree with Newton’s third law?

(d) What would happen if wire 1’s current was in the opposite direction compared to wire 2’s?

11 (a) In the photo of the vacuum tube apparatus in figure o on page 684, infer the direction of the magnetic field from the motion of the electron beam. (The answer is given in the answer to the self-check on that page.)

(b) Based on your answer to part a, find the direction of the currents in the coils.

(c) What direction are the electrons in the coils going?

(d) Are the currents in the coils repelling the currents consisting of the beam inside the tube, or attracting them? Check your answer by comparing with the result of problem 10.

12 A charged particle of mass \( m \) and charge \( q \) moves in a circle due to a uniform magnetic field of magnitude \( B \), which points perpendicular to the plane of the circle.

(a) Assume the particle is positively charged. Make a sketch showing the direction of motion and the direction of the field, and show that the resulting force is in the right direction to produce circular motion.

(b) Find the radius, \( r \), of the circle, in terms of \( m, q, v, \) and \( B \).

(c) Show that your result from part b has the right units.

(d) Discuss all four variables occurring on the right-hand side of your answer from part b. Do they make sense? For instance, what should happen to the radius when the magnetic field is made stronger? Does your equation behave this way?

(e) Restate your result so that it gives the particle’s angular frequency, \( \omega \), in terms of the other variables, and show that \( v \) drops out.

Remark: A charged particle can be accelerated in a circular device called a cyclotron, in which a magnetic field is what keeps them from going off straight. This frequency is therefore known as the cyclotron frequency. The particles are accelerated by other forces (electric forces), which are AC. As long as the electric field is operated at the correct cyclotron frequency for the type of particles being manipulated, it will stay in sync with the particles, giving them a shove in the right direction each time they pass by. The particles are speeding up, so this only works because the cyclotron frequency is independent of velocity.

13 Each figure represents the motion of a positively charged particle. The dots give the particles’ positions at equal time intervals. In each case, determine whether the motion was caused by an electric force, a magnetic force, or a frictional force, and explain
your reasoning. If possible, determine the direction of the magnetic or electric field. All fields are uniform. In (a), the particle stops for an instant at the upper right, but then comes back down and to the left, retracing the same dots. In (b), it stops on the upper right and stays there.

Problem 13.

14 One model of the hydrogen atom has the electron circling around the proton at a speed of $2.2 \times 10^6$ m/s, in an orbit with a radius of 0.05 nm. (Although the electron and proton really orbit around their common center of mass, the center of mass is very close to the proton, since it is 2000 times more massive. For this problem, assume the proton is stationary.) In homework problem 15, p. 567, you calculated the electric current created.

(a) Now estimate the magnetic field created at the center of the atom by the electron. We are treating the circling electron as a current loop, even though it’s only a single particle.

(b) Does the proton experience a nonzero force from the electron’s magnetic field? Explain.

(c) Does the electron experience a magnetic field from the proton? Explain.

(d) Does the electron experience a magnetic field created by its own current? Explain.

(e) Is there an electric force acting between the proton and electron? If so, calculate it.

(f) Is there a gravitational force acting between the proton and electron? If so, calculate it.

(g) An inward force is required to keep the electron in its orbit – otherwise it would obey Newton’s first law and go straight, leaving the atom. Based on your answers to the previous parts, which force or forces (electric, magnetic and gravitational) contributes significantly to this inward force?

[Based on a problem by Arnold Arons.]

15 The equation $B_z = \beta k I A/c^2 r^3$ was found on page 694 for the distant field of a dipole. Show, as asserted there, that the constant